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Karakostas Fixed Point Theorem and Semilinear Neutral Differential Equations with Impulses and Nonlocal Conditions

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Abstract: This paper is concerned with the existence and uniqueness of solutions for a semilinear neutral differential equation with impulses and nonlocal conditions. First, we assume that the nonlinear terms are locally Lipschitz, and to achieve the existence of solutions, Karakostas Fixed Point Theorem is applied. After that, under some additional conditions, the uniqueness is proved as well. Next, assuming some bound on the nonlinear terms the global existence is proved by applying a generalization of Gronwall inequality for impulsive differential equations. Then, we suppose stronger hypotheses on the nonlinear functions, such as globally Lipschitz conditions, that allow us to appy Banach Fixed Point Theorem to prove the existence and uniqueness of solutions. Finally, we present an example as an application of our method.

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1. Introduction and Preliminaries

This work is devoted to study the existence of solutions for the following semilinear neutral differential equation with impulses and nonlocal conditions.

$$\begin{cases} \dfrac{d}{dt}[z(t) - f_{-1}(t, z_t)] = A_0(t)z(t) + f_1(t, z_t), \quad t \neq t_k, \quad t \in [0, \tau], \\ z(\theta) + h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})(\theta) = \eta(\theta), \quad \theta \in [-r, 0], \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), \quad k = 1, 2, \dots, p, \end{cases} \tag{1.1}$$

where $A_0(t)$ is a $n \times n$ continuous matrix, the functions f_{-1}, f_1, and h are smooth enough and $0 < t_1 < t_2 < \cdots < t_p < \tau$, $0 < \tau_1 < \tau_2, \cdots < \tau_q < r < \tau$. Here, $z_t : [-r, 0] \longrightarrow \mathbb{R}^n$ is defined by $z_t(\theta) = z(t + \theta)$, and η belongs to the Banach space

$$\mathcal{PW}_r = \big\{\eta : [-r, 0] \longrightarrow \mathbb{R}^n : \eta \text{ is continuous except at } s_{k\eta}, k = 1, 2, \ldots, p \text{ points} \\ \text{where the side limits exist } \eta(s^+_{k\eta}),\ \eta(s^-_{k\eta}) = \eta(s_{k\eta}), \quad \text{and are finite}\big\}$$

with the norm

$$\|\eta\|_r = \sup_{t \in [-r,0]} \|\eta(t)\|_{\mathbb{R}^n}.$$

There are many papers on the study of linear neutral differential equations, to mention [6, 12–14, 19, 20], particularly, the controllability of such equations has been studied in [12–14, 19, 20] where Kalman-type algebraic condition is proved (see [9]). In [6], the existence of solutions for an abstract neutral functional differential equations is discussed. To our knowledge, there are a few works on the existence of solutions for semilinear neutral equations with impulses and nonlocal conditions simultaneously. Karakostas Fixed Point Theorem will be applied to prove our main result on the existence of solutions of (1.1).

Theorem 1.1 (**Karakostas Fixed Point Theorem- see[7, 10, 11]**). *Let Z and Y be Banach spaces and D be a closed convex subset of Z, and let $\mathcal{B} : D \to Y$ be a continuous operator such that $\mathcal{B}(D)$ is a relatively compact subset of Y, and*

$$\mathcal{T} : D \times \overline{\mathcal{B}(D)} \to D$$

a continuous operator such that the family $\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{B}(D)}\}$ is equicontractive. Then, the operator equation

$$\mathcal{T}(z, \mathcal{B}(z)) = z$$

admits a solution on D.

Now, we define natural Banach spaces where the solutions of problem (1.1) will take place and present some notations to be used through this work. We begin defining the Banach spaces

$$\mathcal{PW}_{t_1..t_p}([0,\tau];\mathbb{R}^n) = \{z : [0,\tau] \to \mathbb{R}^n : z \text{ is continuous except at } t_k, k = 1, \ldots, p \\ \text{points where the side limits exist } z(t^+_k),\ z(t_k) = z(t^-_k), \\ \text{and are finite}\},$$

and

$$\mathcal{PW}_p = \big\{\eta : [-r,\tau] \longrightarrow \mathbb{R}^n : \eta\Big|_{[-r,0]} \in \mathcal{PW}_r \text{ and } \eta\Big|_{[0,\tau]} \in \mathcal{PW}_{t_1..t_p}\big\},$$

equipped with the supremum norm and

$$\|\eta\|_p = \sup_{t \in [-r,\tau]} \|\eta(t)\|_{\mathbb{R}^n},$$

respectively. We will also consider

$$\mathbb{R}^{qn} = \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{q-\text{times}} = \prod_{k=1}^{q} \mathbb{R}^n$$

equipped with the norm

$$\|y\|_q = \sum_{i=1}^{q} \|y_i\|_{\mathbb{R}^n}.$$

Analogously, we define the Banach space

$$\begin{aligned}\mathcal{PW}_{qp} = \big\{\eta : [-r,0] \longrightarrow \mathbb{R}^{qn} : \eta \text{ is continuous except at} s_{k\eta}, k = 1,2,\ldots,p, \text{ points} \\ \text{where the side limits exist } \eta(s_{k\eta}^{+}),\ \eta(s_{k\eta}^{-}) = \eta(s_{k\eta}), \quad \text{and are finite}\big\}\end{aligned}$$

endowed with the norm

$$\|\eta\|_{qp} = \sup_{t\in[-r,0]} \|\eta(t)\|_q = \sup_{t\in[-r,0]} \left(\sum_{i=1}^{q} \|\eta_i(t)\|_{\mathbb{R}^n} \right).$$

The functions in system (1.1) are defined as follows:

$$f_{-1}, f_1 : [0,\tau] \times \mathcal{PW}_r \longrightarrow \mathbb{R}^n, \quad h : \mathcal{PW}_{qp} \longrightarrow \mathcal{PW}_r, \quad J_k : [0,\tau] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

To conclude this section, we define the evolution operator $\mathcal{U}(t,\theta) = \Phi(t)\Phi^{-1}(\theta)$ where Φ is the fundamental matrix of the linear system of ordinary differential equations

$$y'(t) = A_0(t)y(t).$$

Also, we shall consider the following bound

$$M = \sup_{t,\theta\in[0,\tau]} \|\mathcal{U}(t,\theta)\|.$$

Remark 1.1. We will omit the subscript in the functions space norms defined above as long as this does not lead to confusion.

2. Formula for the solutions of system (1.1).

We devote this section to find a formula for solutions of the semilinear neutral differential equations with impulses and nonlocal conditions (1.1). Specifically, we transform problem (1.1) into an integral differential equation problem, which allows us to apply Karakostas Fixed Point Theorem to prove the existence of solutions for (1.1) in the next section.

Proposition 2.1. *The system* (1.1) *has solution* z *on* $[-r,\tau]$ *if, and only if,* z *is a solution of the following integral equation*

$$z(t)=\begin{cases}\mathcal{U}(t,0)\left[\eta(0)-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(0)-f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q}))\right]\\ +\displaystyle\int_0^t\mathcal{U}(t,\theta)\left[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)\right]d\theta+f_{-1}(t,z_t)\\ +\displaystyle\sum_{0<t_k<t}\mathcal{U}(t,t_k)J_k(t_k,z(t_k)),\quad t\in[0,\tau],\\ \eta(t)-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(t),\quad t\in[-r,0].\end{cases}\tag{2.1}$$

Proof. ($\Longrightarrow$) Suppose that z is a solution for system (1.1) on $[-r,\tau]$. Let

$$z_0=\eta(0)-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})(0).$$

• On $[0,t_1)$, z is the solution of the following system

$$\begin{cases}\dfrac{d}{dt}[z(t)-f_{-1}(t,z_t)]=A_0(t)z(t)+f_1(t,z_t),\quad t\in[0,t_1),\\ z(t)+h(z_{\tau_1},\cdots,z_{\tau_q})(t)=\eta(t),\quad t\in[-r,0],\end{cases}$$

and by the variation of parameters formula

$$\begin{aligned}z(t)=&f_{-1}(t,z_t)+\mathcal{U}(t,0)[z_0-f_{-1}\left(0,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})\right)]\\ &+\int_0^t\mathcal{U}(t,\theta)[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)]d\theta,\quad t\in[0,t_1).\end{aligned}$$

As $t\to t_1^-$,

$$\begin{aligned}z(t_1^-)=&f_{-1}(t_1,z_{t_1})+\mathcal{U}(t_1,0)[z_0-f_{-1}\left(0,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})\right)]\\ &+\int_0^{t_1}\mathcal{U}(t_1,\theta)[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)]d\theta.\end{aligned}$$

• On $[t_1,t_2)$, z is the solution of the following system

$$\begin{cases}\dfrac{d}{dt}[z(t)-f_{-1}(t,z_t)]=A_0(t)z(t)+f_1(t,z_t),\quad t\in[t_1,t_2),\\ z(t_1^+)=z(t_1)+J_1(t_1,z(t_1))\end{cases}$$

and again the variation constant formula yields

$$\begin{aligned}z(t)=&f_{-1}(t,z_t)+\mathcal{U}(t,t_1)[z(t_1)+J_1(t_1,z(t_1))-f_{-1}\left(t_1,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q})\right)]\\ &+\int_{t_1}^t\mathcal{U}(t,\theta)[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)]d\theta,\quad t\in[t_1,t_2),\end{aligned}$$

therefore

$$\begin{aligned}
z(t) =& f_{-1}(t, z_t) + \mathcal{U}(t, t_1)\big\{ f_{-1}(t_1, z_{t_1}) + \mathcal{U}(t_1, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t_1} \mathcal{U}(t_1, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + J_1(t_1, z(t_1)) \\
&- f_{-1}\left(t_1, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right) \big\} + \int_{t_1}^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta. \\
=& f_{-1}(t, z_t) + \mathcal{U}(t, t_1)\big\{ \mathcal{U}(t_1, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t} \mathcal{U}(t_1, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + J_1(t_1, z(t_1)) \big\} \\
&+ \int_{t_1}^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta.
\end{aligned}$$

Using the cocycle property of $\mathcal{U}$,

$$\begin{aligned}
z(t) =& f_{-1}(t, z_t) + \mathcal{U}(t, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t_1} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + \mathcal{U}(t, t_1) J_1(t_1, z(t_1)) \\
&+ \int_{t_1}^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta \\
=& f_{-1}(t, z_t) + \mathcal{U}(t, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + \mathcal{U}(t, t_1) J_1(t_1, z(t_1)).
\end{aligned}$$

Proceeding inductively as above, we have that for $t \in [t_p, t_{p+1})$,

$$\begin{aligned}
z(t) =& f_{-1}(t, z_t) + \mathcal{U}(t, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + \sum_{k=1}^{p} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), \quad t \in [0, \tau] \\
=& f_{-1}(t, z_t) + \mathcal{U}(t, 0)[\eta(0) - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0) - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})\right)] \\
&+ \int_0^{t} \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)]d\theta + \sum_{0 < t_k < t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), \quad t \in [0, \tau]
\end{aligned}$$

($\Longleftarrow$) Assume that z is solution of the integral equation (2.1).

Then, at t_1,

$$\begin{aligned} z(t_1^-) =& f_{-1}(t_1, z_{t_1}) + \mathcal{U}(t_1, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})\right)] \\ &+ \int_0^{t_1} \mathcal{U}(t_1, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta, \end{aligned}$$

$$\begin{aligned} z(t_1^+) =& f_{-1}(t_1, z_{t_1}) + \mathcal{U}(t_1, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})\right)] \\ &+ \int_0^{t_1} \mathcal{U}(t_1, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + \mathcal{U}(t_1, t_1) J_1(t_1, z(t_1)), \end{aligned}$$

which implies that

$$z(t_1^+) = z(t_1^-) + J_1(t_1, z(t_1)).$$

Near t_2,

$$\begin{aligned} z(t_2^-) =& f_{-1}(t_2, z_{t_2}) + \mathcal{U}(t_2, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})\right)] \\ &+ \int_0^{t_2} \mathcal{U}(t_2, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + \mathcal{U}(t_2, t_1) J_1(t_1, z(t_1)), \end{aligned}$$

$$\begin{aligned} z(t_2^+) =& f_{-1}(t_2, z_{t_2}) + \mathcal{U}(t_2, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})\right)] \\ &+ \int_0^{t_2} \mathcal{U}(t_2, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + \mathcal{U}(t_2, t_1) J_1(t_1, z(t_1)) \\ &+ \mathcal{U}(t_2, t_2) J_2(t_2, z(t_2)), \end{aligned}$$

which means that

$$z(t_2^+) = z(t_2^-) + J_2(t_2, z(t_2)).$$

Proceeding inductively as above, we get that for $k = 1, 2, \dots, p$,

$$z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)).$$

On the other hand, differentiating z with respect to t, for $t \in [0, \tau)$ and $t \neq t_k, k = 1, 2, \dots, p$, we obtain that

$$\begin{aligned} \frac{d}{dt}(z(t)) =& \frac{d}{dt}\Bigg(f_{-1}(t, z_t) + \mathcal{U}(t, 0)\left[z_0 - f_{-1}(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q}))\right] \\ &+ \int_0^t \mathcal{U}(t, \theta)\left[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)\right] d\theta + \sum_{0<t_k<t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k))\Bigg), \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(z(t)) =& \frac{d}{dt} f_{-1}(t, z_t) + A_0(t)\mathcal{U}(t, 0)[z_0 - f_{-1}\left(0, \eta - h(z_{\tau_1}, z_{\tau_2}, \dots, z_{\tau_q})\right)] \\ &+ A_0(t) \int_0^t \mathcal{U}(t, \theta)[A_0(\theta) f_{-1}(\theta, z_\theta) + f_1(\theta, z_\theta)] d\theta + A_0(t) f_{-1}(t, z_t) + f_1(t, z_t) \\ &+ A_0(t) \sum_{0<t_k<t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)). \end{aligned}$$

By rearranging terms it follows that

$$\begin{aligned}\frac{d}{dt}\left[z(t)-f_{-1}(t,z_t)\right] =& A_0(t)\Big\{ f_{-1}(t,z_t)+\mathcal{U}(t,0)\left[z_0-f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\dots,z_{\tau_q}))\right] \\ &+\int_0^t \mathcal{U}(t,\theta)\left[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)\right]d\theta \\ &+\sum_{0<t_k<t}\mathcal{U}(t,t_k)J_k(t_k,z(t_k))\Big\}+f_1(t,z_t) \\ =& A_0(t)z(t)+f_1(t,z_t),\end{aligned}$$

that is to say, z is a solution of (1.1). □

3. Main Theorems

In this section we shall prove our main result about the existence of solutions for the semilinear neutral equation with impulses and nonlocal conditions (1.1) and their behavior.To achieve that, we consider the following hypotheses on the terms involving the system (1.1).

(H1) There exist constants $d_k, L_g, \gamma > 0, k = 1,2,\dots,p$ such that $\forall y,z \in \mathbb{R}^n$, $t \in [0,\tau]$

i. $L_g q M < \gamma + M\sum_{k=1}^{p} d_k < \frac{1}{2}, \quad \|J_k(t,y)-J_k(t,z)\|_{\mathbb{R}^n} \le d_k\|y-z\|_{\mathbb{R}^n}.$

ii. We have that $h(0)\equiv 0$ and

$$\|h(y)(t)-h(v)(t)\|_{\mathbb{R}^n} \le L_g\sum_{i=1}^{q}\|y_i(t)-v_i(t)\|_{\mathbb{R}^n}, \quad y,v\in\mathcal{PW}_{qp}.$$

(H2) The function f_{-1} satisfies

i.

$$\begin{aligned}&\|A_0(t)f_{-1}(t,\eta_1)-A_0(t)f_{-1}(t,\eta_2)\|_{\mathbb{R}^n} \le \mathcal{K}\left(\|\eta_1\|_r,\|\eta_2\|_r\right)\|\eta_1-\eta_2\|_r,\ \eta_1,\eta_2\in\mathcal{PW}_r,\\ &\|f_{-1}(t,\eta_1)-f_{-1}(t,\eta_2)\|_{\mathbb{R}^n} \le \gamma\|\eta_1-\eta_2\|_r, \quad \eta_1,\eta_2\in\mathcal{PW}_r\\ &\|A_0(t)f_{-1}(t,\eta)\|_{\mathbb{R}^n} \le \Psi\left(\|\eta\|_r\right), \quad \eta\in\mathcal{PW}_r,\\ &\|f_{-1}(t,\eta)\|_{\mathbb{R}^n} \le \Psi\left(\|\eta\|_r\right), \quad \eta\in\mathcal{PW}_r.\end{aligned}$$

and f_1 satisfies

ii.

$$\begin{aligned}&\|f_1(t,\eta_1)-f_1(t,\eta_2)\|_{\mathbb{R}^n} \le \mathcal{K}\left(\|\eta_1\|_r,\|\eta_2\|_r\right)\|\eta_1-\eta_2\|_r, \quad \eta_1,\eta_2\in\mathcal{PW}_r,\\ &\|f_1(t,\eta)\|_{\mathbb{R}^n} \le \Psi\left(\|\eta\|_r\right), \quad \eta\in\mathcal{PW}_r,\end{aligned}$$

where $\mathcal{K} : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are continuous and non decreasing functions.

(H3) There exists $\rho, \tau > 0$ such that

$$M\Psi\left(\|\eta\| + L_g q\left(\|\tilde{\eta}\| + \rho\right)\right) + \left(ML_g q + M\sum_{k=1}^{p} d_k\right)\left(\|\tilde{\eta}\| + \rho\right)$$
$$+ (2M\tau + 1)\Psi\left(\|\tilde{\eta}\| + \rho\right) < \rho$$

where the function $\tilde{\eta}$ is defined as follows

$$\tilde{\eta}(t) = \begin{cases} \mathcal{U}(t,0)\eta(0), & t \in [0,\tau], \\ \eta(t), & t \in [-r,0]. \end{cases}$$

(H4) Assume the following relation holds

$$M\left\{L_g q\,(1+\gamma) + 2\tau\mathcal{K}\left(\|\tilde{\eta}\| + \rho, \|\tilde{\eta}\| + \rho\right)\right\} < \frac{1}{2}.$$

Remark 3.1. The hypothesis **(H2)** is not a whim, it appears naturally when one studies the well-known Burgues equation and the Benjamin-Bona-Mahony equation; and since we will extend this work to infinite-dimensional Hilbert spaces, these hypotheses are considered here. For more details about it, one can see [10, 11].

Theorem 3.1. *Suppose that* **(H1)-(H3)** *hold. Then, the system* (1.1) *has at least one solution on* $[-r,\tau]$.

Proof. We shall transform the problem of proving the existence of solutions for system (1.1) into a fixed point problem. For this, we define the following operators

$$\mathcal{T} : \mathcal{PW}_p \times \mathcal{PW}_p \longrightarrow \mathcal{PW}_p,$$

and

$$\mathcal{B} : \mathcal{PW}_p \longrightarrow \mathcal{PW}_p$$

given by

$$\mathcal{T}(z,y)(t) = \begin{cases} y(t) + f_{-1}(t,z_t) + \sum_{0<t_k<t} \mathcal{U}(t,t_k)J_k(t_k,z(t_k)), & t \in [0,\tau], \\ \eta(t) - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(t), & t \in [-r,0], \end{cases}$$

and

$$\mathcal{B}(y)(t) = \begin{cases} \mathcal{U}(t,0)\left[\eta(0) - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0) - f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}))\right] \\ + \int_0^t \mathcal{U}(t,\theta)\left[A_0(\theta)f_{-1}(\theta,y_\theta) + f_1(\theta,y_\theta)\right]d\theta, \quad t \in [0,\tau], \\ \eta(t), \quad t \in [-r,0], \end{cases}$$

respectively. We also consider the following closed and convex set

$$D = D(\rho, \tau, \eta) = \{y \in \mathcal{PW}_p : \|y - \tilde{\eta}\|_p \leq \rho\}.$$

With this setting, the problem of finding solutions for system (1.1) has been reduced to the problem of finding solutions of the following operator equation

$$\mathcal{T}(z, \mathcal{B}(z)) = z.$$

The rest of the proof will be given by statements as follows:

Statement 1. *$\mathcal{B}$ is a continuous mapping.*

For any $z, y \in \mathcal{PW}_p$ we have that

$$\begin{aligned}
\|\mathcal{B}(z)(t) - \mathcal{B}(y)(t)\| \ &\leq \|\mathcal{U}(t,0)\| \Big\{ \big\| h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0) - h(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q})(0) \big\| \\
&\quad + \big\| f_{-1}\big(0, \eta - h(y_{\tau_1}, y_{\tau_2}, ..., y_{\tau_q})\big) - f_{-1}\big(0, \eta - h(z_{\tau_1}, z_{\tau_2}, ..., z_{\tau_q})\big) \big\| \Big\} \\
&\quad + \int_0^t \|\mathcal{U}(t,\theta)\| \Big\{ \|A_0(\theta) f_{-1}(\theta, z_\theta) - A_0(\theta) f_{-1}(\theta, y_\theta)\| \\
&\quad + \|f_1(\theta, z_\theta) - f_1(\theta, y_\theta)\| \Big\} d\theta \\
&\leq M \left[L_g q \|z - y\| + \gamma \big\| g(z_{\tau_1}, z_{\tau_2}, \ldots, z_{\tau_q}) - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}) \big\| \right] \\
&\quad + M\tau \left[\mathcal{K}(\|z\|, \|y\|) \|z - y\| + \mathcal{K}(\|z\|, \|y\|) \|z - y\| \right], \\
&\leq M \left[L_g q \|z - y\| + \gamma L_g q \|z - y\| \right] \\
&\quad + 2M\tau \mathcal{K}(\|z\|, \|y\|) \|z - y\|,
\end{aligned}$$

where the last two inequality comes from **(H1-ii)** and **(H2)**. It follows that

$$\|\mathcal{B}(z) - \mathcal{B}(y)\| \ \leq M \left\{ L_g q \,(1 + \gamma) + 2\tau \mathcal{K}\left(\|z\|, \|y\|\right) \right\} \|z - y\|$$

by taking supremum over $t \in [-r, \tau]$. Hence $\mathcal{B}$ is locally Lipschitz, which implies the continuity of $\mathcal{B}$.

Statement 2. *$\mathcal{B}$ maps bounded sets of $\mathcal{PW}_p$ into bounded sets of $\mathcal{PW}_p$.*

In order to prove this statement, we will show that

$$\forall R > 0 \ \exists \lambda > 0 \ \forall y \in B_R : \|\mathcal{B}(y)\| \leq \lambda,$$

where $B_R = \{z \in \mathcal{PW}_p : \|z\| \leq R\}$. Let $R > 0$ and consider $\lambda = \max\{\vartheta, \|\eta\|\}$, ϑ to be determined later. Let $y \in B_R$. Then, on one hand, we have that

$$\|\mathcal{B}(y)(t)\| = \|\eta(t)\| \leq \|\eta\|,$$

if $t \in [-r, 0]$. While, on the other hand,

$$\begin{aligned}
\|\mathcal{B}(y)(t)\| &\le \|\mathcal{U}(t,0)\| \big\| \eta(0) - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0) - f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})) \big\| \\
&\quad + \int_0^t \|\mathcal{U}(t,\theta)\| \left[\|A_0(\theta) f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\| \right] d\theta \\
&\le M \big\{ \|\eta(0)\| + \big\| h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0) \big\| + \big\| f_{-1}(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})) \big\| \big\} \\
&\quad + \tau M \left[\|A_0(\theta) f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\| \right] \\
&\le M \big\{ \|\eta(0)\| + L_g q \|y\| + \Psi \left(\big\| \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}) \big\| \right) \big\} + \tau M 2 \Psi(\|y\|) \\
&\le M \big\{ \|\eta(0)\| + L_g q \|y\| + \Psi \left(\|\eta\| + \big\| h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}) \big\| \right) \big\} + \tau M 2 \Psi(\|y\|) \\
&\le M \big\{ \|\eta(0)\| + L_g q \|y\| + \Psi \left(\|\eta\| + L_g q \|y\| \right) \big\} + \tau M 2 \Psi(\|y\|) \\
&\le M \big\{ \|\eta(0)\| + L_g q R + \Psi \left(\|\eta\| + L_g q R \right) + \tau 2 \Psi(R) \big\} = \vartheta,
\end{aligned}$$

if $t \in [0, \tau]$. Here we have used **(H1-ii)** and **(H2)**. Now, taking supremum over $t \in [-r, \tau]$, we have that

$$\|\mathcal{B}(y)\| \le \lambda.$$

Statement 3. *$\mathcal{B}$ maps bounded sets of $\mathcal{PW}_p$ into equicontinuous sets of $\mathcal{PW}_p$.*

Let us consider B_R as above and let us show that $\mathcal{B}(B_R)$ is equicontinuous on $[-r, \tau]$. On $[-r, 0]$, the continuity of η immediately implies the result. On $(0, \tau]$, we have that

$$\begin{aligned}
\|\mathcal{B}(y)(t_2) - \mathcal{B}(y)(t_1)\| &\le \|\mathcal{U}(t_2, 0) - \mathcal{U}(t_1, 0)\| \|\eta(0) - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q})(0) \\
&\quad - f_{-1} \left(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}) \right) \| \\
&\quad + \int_0^{t_1} \|\mathcal{U}(t_2, \theta) - \mathcal{U}(t_1, \theta)\| \|A_0(\theta) f_{-1}(\theta, y_\theta) + f_1(\theta, y_\theta)\| d\theta \\
&\quad + \int_{t_1}^{t_2} \|\mathcal{U}(t_2, \theta)\| \|A_0(\theta) f_{-1}(\theta, y_\theta) + f_1(\theta, y_\theta)\| d\theta \\
&\le \|\mathcal{U}(t_2, 0) - \mathcal{U}(t_1, 0)\| \Big\{ \|\eta(0)\| + L_g q \|y\| \\
&\quad + \big\| f_{-1} \left(0, \eta - h(y_{\tau_1}, y_{\tau_2}, \ldots, y_{\tau_q}) \right) \big\| \Big\} \\
&\quad + \int_0^{t_1} \|\mathcal{U}(t_2, \theta) - \mathcal{U}(t_1, \theta)\| \left[\|A_0(\theta) f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\| \right] d\theta \\
&\quad + \int_{t_1}^{t_2} \|\mathcal{U}(t_2, \theta)\| \left[\|A_0(\theta) f_{-1}(\theta, y_\theta)\| + \|f_1(\theta, y_\theta)\| \right] d\theta \\
&\le \|\mathcal{U}(t_2, 0) - \mathcal{U}(t_1, 0)\| \Big\{ \|\eta(0)\| + L_g q \|y\| + \Psi \left(\|\eta\| + L_g q \|y\| \right) \Big\} \\
&\quad + 2\Psi \left(\|y\| \right) \int_0^{t_1} \|\mathcal{U}(t_2, \theta) - \mathcal{U}(t_1, \theta)\| d\theta + 2M \Psi \left(\|y\| \right) (t_2 - t_1)
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\mathcal{U}(t_2,0)-\mathcal{U}(t_1,0)\|\left\{\|\eta(0)\|+L_g qR+\Psi\left(\|\eta\|+L_g qR\right)\right\}\\
&\quad+2\Psi\left(R\right)\int_0^{t_1}\|\mathcal{U}(t_2,\theta)-\mathcal{U}(t_1,\theta)\|d\theta+2M\Psi\left(R\right)(t_2-t_1)\rightarrow 0
\end{aligned}
$$

as $t_2 \to t_1$ by the continuity of $\mathcal{U}$ and the fact that $\|\eta(0)\| + L_g qR + \Psi\left(\|\eta\| + L_g qR\right)$ is bounded. Here we have considered **(H1-ii)** and **(H2)**. This shows that $\mathcal{B}(B_R)$ is equicontinuous.

Statement 4. *The subset $\mathcal{B}(D)$ is relatively compact in $\mathcal{PW}_p$.*

Let us prove Statement 4. Let D be a bounded subset of $\mathcal{PW}_p$. By Statements 2 and 3, $\mathcal{B}(D)$ is bounded and equicontinuous in $\mathcal{PW}_p$. Let $\{y_n\}_{n\in\mathbb{N}} \subseteq \mathcal{B}(D)$, then

$$
y_n\Big|_{[-r,0]} = \eta, \ \forall n \in \mathbb{N}.
$$

Hence, $y_n\Big|_{[-r,0]}$ converges uniformly on $[-r, 0]$.

Now, putting $\varphi_n = y_n\Big|_{[0,\tau]}$, we get that $\{\varphi_n\}_{n\in\mathbb{N}} \subseteq \mathcal{PW}_{t_1..t_p}$.
Let us put $t_0 = 0$ and $t_{p+1} = \tau$. Then, applying Arzela-Ascoli Theorem, the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ contains a subsequence $\{\varphi_n^1\}_{n\in\mathbb{N}}$ that converges in the interval $[t_0, t_1]$. Now, applying Arzela-Ascoli Theorem again, we get that the sequence $\{\varphi_n^1\}_{n\in\mathbb{N}}$ contains a subsequence $\{\varphi_n^2\}_{n\in\mathbb{N}}$ that converges in the interval $[t_1, t_2]$. Continuing with this process we find a subsequence $\{\varphi_n^{p+1}\}_{n\in\mathbb{N}}$ of $\{\varphi_n\}_{n\in\mathbb{N}}$ that converges in each interval $[t_k, t_{k+1}]$, with $k = 0, 1, 2, \ldots, p$. Therefore,

$$
\varphi_n^{p+1} = y_n^{p+1}\Big|_{[0,\tau]} \text{ converges on } [0, \tau].
$$

Consequently, $\{\varphi_n^{p+1}\}_{n\in\mathbb{N}} = \{y_n^{p+1}\}_{n\in\mathbb{N}}$ converges uniformly on $[-r, \tau]$. Thus, $\mathcal{B}(D)$ is relatively compact, and the proof of Statement 4 is completed.

Statement 5. *The family $\left\{\mathcal{T}(\cdot, y) : y \in \overline{\mathcal{B}(D)}\right\}$ is equicontractive.*

On the one hand, for any $u, v \in \mathcal{PW}_p$ and $t \in [-r, 0]$, we get that

$$
\begin{aligned}
\|\mathcal{T}(u,\mathcal{B}(y))(t)-\mathcal{T}(v,\mathcal{B}(y))(t)\| &\leq \left\|h(u_{\tau_1},u_{\tau_2},\ldots,u_{\tau_q})(t)-h(v_{\tau_1},v_{\tau_2},\ldots,v_{\tau_q})(t)\right\|\\
&\leq L_g q\|u-v\|\\
&\leq ML_g q\|u-v\|.
\end{aligned}
$$

While on the other hand, by using **(H1-i)** and **(H2-i)**, for all $t \in (0, \tau]$ we obtain that

$$\begin{aligned}\|\mathcal{T}(u,\mathcal{B}(y))(t)-\mathcal{T}(v,\mathcal{B}(y))(t)\| &\le \|f_{-1}(t,u_t)-f_{-1}(t,v_t)\|\\ &\quad+\sum_{0<t_k<t}\|\mathcal{U}(t,t_k)\left[J_k(t_k,u(t_k))-J_k(t_k,v(t_k))\right]\|\\ &\le \gamma\|u-v\|+M\sum_{k=1}^{p}\|J_k(t_k,u(t_k))-J_k(t_k,v(t_k))\|\\ &\le \gamma\|u-v\|+M\sum_{k=1}^{p}d_k\|u(t_k)-v(t_k)\|\\ &\le \gamma\|u-v\|+M\|u-v\|\sum_{k=1}^{p}d_k\\ &\le \left(\gamma+M\sum_{k=1}^{p}d_k\right)\|u-v\|.\end{aligned}$$

It follows that

$$\|\mathcal{T}(u,\mathcal{B}(y))-\mathcal{T}(v,\mathcal{B}(y))\|\le\left(\gamma+M\sum_{k=1}^{p}d_k\right)\|u-v\|\le\frac{1}{2}\|u-v\|$$

by taking supremum over $t\in[-r,\tau]$ and using **(H1-i)**. This shows that $\mathcal{T}(\cdot,\mathcal{B}(y))$ is a contraction which does not depend on $y\in\overline{\mathcal{B}(D)}$.

Statement 6. *The inclusion* $\mathcal{T}\left(\cdot,\mathcal{B}(\cdot)\right)\left(D(\rho,\tau,\eta)\right)\subset D(\rho,\tau,\eta)$ *holds.*

Let $z\in D(\rho,\tau,\eta)$ be arbitrary. Notice that

$$\mathcal{T}(z,\mathcal{B}(z))(t)=\begin{cases}\mathcal{U}(t,0)\big[\eta(0)-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q})(0)-f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q}))\big]\\ +\displaystyle\int_0^t\mathcal{U}(t,\theta)\left[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)\right]d\theta+f_{-1}(t,z_t)\\ +\displaystyle\sum_{0<t_k<t}\mathcal{U}(t,t_k)J_k(t_k,z(t_k)), & t\in[0,\tau],\\ \eta(t)-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q})(t), & t\in[-r,0].\end{cases}$$

On the one hand, for $t\in[-r,0]$, we have that

$$\begin{aligned}\|\mathcal{T}(z,\mathcal{B}(z))(t)-\tilde{\eta}(t)\| &\le \left\|g\left(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q}\right)(t)\right\|\\ &\le L_gq\|z\|\\ &\le ML_gq\|z\|\\ &\le ML_gq\left(\|\tilde{\eta}\|+\rho\right)\\ &<\rho.\end{aligned}$$

While on the other hand, for $t \in [0,\tau]$, we have that

$$\begin{aligned}
\|\mathcal{T}(z,\mathcal{B}(z))(t)-\tilde{\eta}(t)\| &\le M\big\|h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q})(0)-f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q}))\big\| \\
&\quad+\int_0^t \|\mathcal{U}(t,\theta)\left[A_0(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)\right]\|d\theta+\|f_{-1}(t,z_t)\| \\
&\quad+\sum_{0<t_k<t}\|\mathcal{U}(t,t_k)J_k(t_k,z(t_k))\| \\
&\le M\left\{L_g q\|z\|+\big\|f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q}))\big\|\right\} \\
&\quad+2M\tau\Psi(\|z\|)+\Psi(\|z\|)+M\sum_{0<t_k<t}\|J_k(t_k,z(t_k))\| \\
&\le M\left\{L_g q\|z\|+\Psi\left(\|\eta\|+L_g q\|z\|\right)\right\} \\
&\quad+2M\tau\Psi(\|z\|)+\Psi(\|z\|)+\left(M\sum_{k=1}^{p}d_k\right)\|z\| \\
&\le M\left\{L_g q\left(\|\tilde{\eta}\|+\rho\right)+\Psi\left(\|\tilde{\eta}\|+L_g q\left(\|\tilde{\eta}\|+\rho\right)\right)\right\} \\
&\quad+2M\tau\Psi\left(\|\tilde{\eta}\|+\rho\right)+\Psi\left(\|\tilde{\eta}\|+\rho\right)+\left(M\sum_{k=1}^{p}d_k\right)\left(\|\tilde{\eta}\|+\rho\right) \\
&\le M\Psi\left(\|\tilde{\eta}\|+L_g q\left(\|\tilde{\eta}\|+\rho\right)\right)+\left(ML_g q+M\sum_{k=1}^{p}d_k\right)\left(\|\tilde{\eta}\|+\rho\right) \\
&\quad+(2M\tau+1)\Psi\left(\|\tilde{\eta}\|+\rho\right)<\rho.
\end{aligned}$$

Here we have used **(H3)**. Now, by taking supremum over $t \in [-r,\tau]$, we get that

$$\|\mathcal{T}(z,\mathcal{B}(z))-\tilde{\eta}\| \le \rho.$$

and by Karakostas Fixed Point Theorem the operator equation

$$\mathcal{T}(z,\mathcal{B}(z))=z$$

admits a solution on D. This finishes the proof. □

Theorem 3.2. *System* (1.1) *has a unique solution if* ***(H4)*** *is additionally assumed.*

Proof. Suppose u and v are two solutions of system (1.1). Now, considering **(H1)** and **(H2)** we have that

$$\begin{aligned}
\|u(t)-v(t)\| \le \|\mathcal{U}(t,0)\|&\Big\{\big\|h(u_{\tau_1},u_{\tau_2},\ldots,u_{\tau_q})(0)-h(v_{\tau_1},v_{\tau_2},\ldots,v_{\tau_q})(0)\big\| \\
&+\big\|f_{-1}\left(0,\eta-h(u_{\tau_1},u_{\tau_2},\ldots,u_{\tau_q})\right)-f_{-1}\left(0,\eta-h(v_{\tau_1},v_{\tau_2},\ldots,v_{\tau_q})\right)\big\|\Big\} \\
&+\int_0^t\|\mathcal{U}(t,\theta)\|\Big\{\|A_0(\theta)f_{-1}(\theta,u_\theta)-A_0(\theta)f_{-1}(\theta,v_\theta)\|+\|f_1(\theta,u_\theta)-f_1(\theta,v_\theta)\|\Big\}d\theta
\end{aligned}$$

$$+\|f_{-1}(t,u_t)-f_{-1}(t,v_t)\|+\sum_{0<t_k<t}\|\mathcal{U}(t,t_k)\|\|J_k(t_k,u(t_k))-J_k(t_k,v(t_k))\|$$

$$\leq M\left\{L_g q\,(1+\gamma)+2\tau\mathcal{K}\left(\|u\|,\|v\|\right)\right\}\|u-v\|+\left(\gamma+M\sum_{k=1}^{p}d_k\right)\|u-v\|$$

$$\leq M\left\{L_g q\,(1+\gamma)+2\tau\mathcal{K}\left(\|\tilde{\eta}\|+\rho,\|\tilde{\eta}\|+\rho\right)\right\}\|u-v\|+\frac{1}{2}\|u-v\|$$

Bearing in mind the hypothesis **(H4)**, and taking supremum over $t\in[-r,\tau]$, we have that

$$\|u-v\|\leq\omega\|u-v\|$$

with $0\leq\omega<1$. This implies $\|u-v\|=0$, and therefore $u=v$. □

Next, we consider the following subset $\tilde{D}$ of $\mathbb{R}^n$:

$$\tilde{D}=\{v\in\mathbb{R}^n:\|v\|_{\mathbb{R}^n}\leq\rho\}. \tag{3.1}$$

Therefore, for all $y\in D$ we have $y(t)-\tilde{\eta}(t)\in\tilde{D}$ for $t\in[-r,\tau]$.

Definition 3.1. We shall say that $[-r,\theta_1)$ is a maximal interval of existence for the solution z of problem (1.1) if there is not solution of (1.1) on $[-r,\theta_2)$ with $\theta_2>\theta_1$.

Theorem 3.3. *Suppose that the conditions of Theorem 3.1 hold. If z is a solution of problem (1.1) on $[-r,\theta_1)$ and θ_1 is maximal, then either $\theta_1=+\infty$ or there exists a sequence $\tau_n\to\theta_1$ as $n\to\infty$ such that $z(\tau_n)-\tilde{\eta}(\tau_n)\to\partial\tilde{D}$.*

Proof. Suppose $\theta_1<\infty$. For the purpose of contradiction assume the existence of a neighborhood N of $\partial\tilde{D}$ such that $\{z(t)-\tilde{\eta}(t)\}$ does not enter in it, for $0<\theta_2\leq t<\theta_1$. We can take $N=\tilde{D}\backslash B$, where B is a closed subset of $\tilde{D}$, then $z(t)-\tilde{\eta}(t)\in B$ for $0<t_p<\theta_2\leq t<\theta_1$. We need to prove that $\lim\limits_{t\to\theta_1^-}\{z(t)-\tilde{\eta}(t)\}=z_1-\tilde{\eta}(\theta_1)\in B$, which is enough to prove that $\lim\limits_{t\to\theta_1^-}z(t)=z_1$. Indeed, if we consider $0<t_p<\theta_2\leq\ell<t<\theta_1$, then:

$$\begin{aligned}\|z(t)-z(\ell)\|\leq&\|\mathcal{U}(t,0)-\mathcal{U}(\ell,0)\|\big(\|\eta(0)\|+\big\|h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q})(0)\big\|\\&+\big\|f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q}))\big\|\big)\\&+\int_0^{\ell}\|\mathcal{U}(t,\theta)-\mathcal{U}(\ell,\theta)\|\|A(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)\|d\theta\\&+\int_{\ell}^{t}\|\mathcal{U}(t,\theta)\|\|A(\theta)f_{-1}(\theta,z_\theta)+f_1(\theta,z_\theta)\|d\theta+\|f_{-1}(t,z_t)-f_{-1}(\ell,z_\ell)\|\\&+\sum_{0<t_k<\ell}\|\mathcal{U}(t,t_k)-\mathcal{U}(\ell,t_k)\|\|J_k(t_k,z(t_k))\|\\&+\sum_{\ell<t_k<t}\|\mathcal{U}(t,t_k)\|\|J_k(t_k,z(t_k))\|\end{aligned}$$

$$
\begin{aligned}
&\le \|\mathcal{U}(t,0)-\mathcal{U}(\ell,0)\|\big(\|\eta(0)\|+L_g q\|z\|+\Psi\left(\|\eta\|+L_g q\|z\|\right)\big)\\
&\quad +\Big(\int_0^\ell \|\mathcal{U}(t,\theta)-\mathcal{U}(\ell,\theta)\|d\theta+\int_\ell^t \|(\mathcal{U}(t,\theta)\|d\theta\Big)2\Psi(\|z\|)\\
&\quad +\|f_{-1}(t,z_t)-f_{-1}(\ell,z_\ell)\|+\|\mathcal{U}(t,\ell)-I\|\sum_{k=1}^{q}\|\mathcal{U}(\ell,t_k)\|\|J_k(z(t_k)\|\\
&\quad +\sum_{\ell<t_k<t}\|\mathcal{U}(t,t_k)\|\|J_k(t_k,z(t_k))\|\\
&\le \|\mathcal{U}(t,0)-\mathcal{U}(\ell,0)\|\big(\|\eta(0)\|+L_g q\|z\|+\Psi\left(\|\eta\|+L_g q\|z\|\right)\big)\\
&\quad +\Big(\int_0^\ell \|\mathcal{U}(t,\theta)-\mathcal{U}(\ell,\theta)\|d\theta+\int_\ell^t \|(\mathcal{U}(t,\theta)\|d\theta\Big)2\Psi(\|z\|)\\
&\quad +\|f_{-1}(t,z_t)-f_{-1}(\ell,z_\ell)\|+\|\mathcal{U}(t,\ell)-I\|M\sum_{k=1}^{q}\|J_k(z(t_k)\|\\
&\quad +\sum_{\ell<t_k<t}\|\mathcal{U}(t,t_k)\|\|J_k(t_k,z(t_k))\|
\end{aligned}
$$

Since $\mathcal{U}$ is uniformly continuous for $t\ge 0$, then $\|z(t)-z(l)\|_{\mathbb{R}^n}$ goes to zero as $l\to\theta_1^-$. Therefore, $\lim_{t\to\theta_1^-} z(t)=z_1$ exists in $\mathbb{R}^n$ and, since B is closed, $z_1-\tilde{\eta}(\theta_1)$ belongs to B. This will contradict the maximality of θ_1. In fact, we have that $z_1\in B+\tilde{\eta}(\theta_1)$ is contained in the interior of the ball $\tilde{D}+\tilde{\eta}(\theta_1)$. Hence, $z(\cdot)$ can be extended to $[-r,\theta_1]$. In this regard, for ϵ small enough, the following initial value problem admit only one solutions on $[-r,\theta_1+\epsilon)$

$$
\begin{cases}
\dfrac{d}{dt}[u(t)-f_{-1}(t,u_t)]=A_0(t)u(t)+f_1(t,u_t), \quad t\in[\theta_1,\theta_1+\epsilon),\\
u(\theta)=z(\theta), \quad \theta\in[\theta_1-r,\theta_1].
\end{cases}
\tag{3.2}
$$

This is a contradiction with the maximality of θ_1. So, the proof is completed. □

Corollary 3.1. *In the conditions of Theorem 3.1, if the second part of hypothesis* **(*H*1)** *is changed to*

$$
\|f_1(t,\eta)\|\le\mu(t)(1+\|\eta(0)\|_{\mathbb{R}^n}), \quad \eta\in\mathcal{PW}_r, \quad t\in[-r,\infty),
$$

where μ is a continuous function on $[-r,\infty)$, then a unique solution of problem (1.1) exists on $[-r,\infty)$.

Proof.

$$
\begin{aligned}
\|z(t)\| &\le \|\mathcal{U}(t,0)\|\|\eta(0)-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q})(0)-f_{-1}(0,\eta-h(z_{\tau_1},z_{\tau_2},\ldots,z_{\tau_q}))\|\\
&\quad +M\int_0^t \|A_0(\theta)f_{-1}(\theta,z_\theta)\|+\|f_1(\theta,z_\theta)\|d\theta+\|f_{-1}(t,z_t)\|\\
&\quad +\sum_{0<t_k<t} M\|J_k(t_k,z(t_k))\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|\mathcal{U}(t,0)\| \big(\|\eta(0)\| + L_g q \|z\| + \Psi\left(\|\eta\| + L_g q \|z\| \right) \big) \\
&\quad + M \int_0^t \|A_0(\theta) f_{-1}(\theta, z_\theta)\| + \mu(\theta)(1 + \|z(\theta)\|) d\theta + \|f_{-1}(t, z_t)\| \\
&\quad + \sum_{0<t_k<t} M d_k \|z(t_k)\|.
\end{aligned}
$$

Then, applying Gronwall Inequality for impulsive differential equations(see [8, 15, 16, 18]), we obtain that

$$
\|z(t)\|_{\mathbb{R}^n} \leq M \left(\|z(0)\|_{\mathbb{R}^n} + \int_0^\tau \mu(\theta) d\theta \right) \prod_{t_0<t_k<t} (1 + M d_k) e^{\int_0^\tau M\mu(\theta) d\theta},
$$

This implies that $\|z(t)\|_{\mathbb{R}^n}$ remains bounded as $t \to \theta_1$ and applying Theorem 3.3 we get the result.

□

4. Global Lipschitz Conditions

This section will assume stronger hypotheses on the nonlinear terms that allow us to apply Banach Fixed Point Theorem. Specifically, we will suppose that the nonlinear functions that appear in our system are globally Lipschitz. Moreover, we shall consider the following simpler system

$$
\begin{cases}
\dfrac{d}{dt}[z(t) - f(t, z_t)] = A_0(t) z(t) + F(t, z_t), \quad t \in [0, \tau] \setminus \{t_1, t_2, \dots, t_p\} \\
z(s) = g(z)(s) + \phi(s), \quad s \in [-r, 0] \\
z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k)), \quad k = 1, 2, \dots, p,
\end{cases}
\tag{4.1}
$$

where the nonlocal condition $z(s) = g(z)(s) + \phi(s), \quad s \in [-r, 0]$ means

$$
z(s) = g\left(z\Big|_{[-r,0]} \right)(s) + \phi(s), \quad s \in [-r, 0].
$$

The functions $f, F : [0, \tau] \times \mathcal{PW}_r \longrightarrow \mathbb{R}^n$ are smooth enough satisfying certain conditions that will be specified later, and $J_k : [0, \infty) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $k = 1, 2, \dots, p$, are continuous and represents the impulsive effect in the system (4.1), the continuous function $g : \mathcal{PW}_r \longrightarrow \mathcal{PW}_r$ represent the nonlocal conditions, this function acts as a feedback operator which adjusts a part of the past when the initial function is present, or even, the whole past when the function ϕ is absent according to some precise future requirements (see [1]). The advantage of using nonlocal conditions is that measurements at more places can be incorporated to get better models. For more details and physical interpretations about nonlocal condition see [1–5, 21] and references therein.

Now, assuming a global Lipschitz condition, we will prove that system (4.1) admits a unique solution defined on $[0, \tau]$ by applying Banach Fixed Point Theorem. In this regards, we suppose the following global Lipschitz condition on the nonlinear terms:

(L1) There exist positive constants L_f and L_F such that for all $t \in [0, \tau]$, $\phi, \tilde{\phi} \in \mathcal{PW}_r$

$$\|f(t,\phi) - f(t,\tilde{\phi})\| \le L_f \|\phi - \tilde{\phi}\|_r,$$
$$\left\|F(t,\phi) - F(t,\tilde{\phi})\right\| \le L_F \|\phi - \tilde{\phi}\|_r.$$

(L2) There exist nonnegative constants d_k, $k = 1, 2, \dots, p$ such that for all $t \in [0, \infty)$, $z, \tilde{z} \in \mathbb{R}^n$

$$\|J_k(t,z) - J_k(t,\tilde{z})\|_{\mathbb{R}^n} \le d_k \|z - \tilde{z}\|_{\mathbb{R}^n}.$$

(L3) There exists a nonnegative constant L_g such that for all $\phi, \psi \in \mathcal{PW}_r$

$$\|g(\phi) - g(\psi)\|_r \le L_g \|\phi - \psi\|_r.$$

(L4)

$$L_f + M[L_g + L_f L_g + \|A_0\| L_f \tau + L_F \tau + \sum_{k=1}^{p} d_k] < 1,$$

where $\|A_0\| = \max\{\|A_0(t)\| : t \in [0, \tau]\}$.

Proposition 4.1. *Let $\phi \in \mathcal{PW}_r$. Then z is solution of system* (4.1) *if and only if z satisfies the integral equation*

$$z(t) = \begin{cases} g(z)(t) + \phi(t), \quad t \in [-r, 0], \\ f(t, z_t) + \mathcal{U}(t,0)\big[g(z)(0) + \phi(0) - f(0, g(z)(0) + \phi(0))\big] \\ + \displaystyle\int_0^t \mathcal{U}(t,s) A_0(s) f(s, z_s) ds \\ + \displaystyle\int_0^t \mathcal{U}(t,s) F(s, z_s) ds + \sum_{0<t_k<t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), \quad t \in [0, \tau], \end{cases} \tag{4.2}$$

Theorem 4.1. *Suppose that* (L1)-(L4) *hold. Then for $\phi \in \mathcal{PW}_r$ the system* (4.1) *has a unique solution defined on* $[0, \tau]$.

Proof. We shall apply Banach Contraction Mapping Theorem, in this regard, we will define the following operator $\mathcal{T} : \mathcal{PW}_p \longrightarrow \mathcal{PW}_p$ by

$$\mathcal{T}(t) = \begin{cases} g(z)(t) + \phi(t), \quad t \in [-r, 0], \\ f(t, z_t) + \mathcal{U}(t,0)\big[g(z)(0) + \phi(0) - f(0, g(z)(0) + \phi(0))\big] \\ + \displaystyle\int_0^t \mathcal{U}(t,s) A_0(s) f(s, z_s) ds \\ + \displaystyle\int_0^t \mathcal{U}(t,s) F(s, z_s) ds + \sum_{0<t_k<t} \mathcal{U}(t, t_k) J_k(t_k, z(t_k)), \quad t \in [0, \tau]. \end{cases} \tag{4.3}$$

If $t \in [-r, 0]$, then

$$\begin{aligned} \|(\mathcal{T}z)(t) - (\mathcal{T}\tilde{z})(t)\| &= \|g(z)(t) - g(\tilde{z})(t)\| \le \|(g(z) - g(\tilde{z}))\Big|_{[-r,0]}\|_p \\ &\le L_g \|(z - \tilde{z})\Big|_{[-r,0]}\|_{\mathcal{PW}_r} \le L_g \|z - \tilde{z}\|_p. \end{aligned}$$

If $t \in [0,\tau]$, then

$$\begin{aligned}\|(\mathcal{T}z)(t)-(\mathcal{T}\tilde{z})(t)\| &\leq \|f(t,z_t)-f(t,\tilde{z}_t)\| + \|\mathcal{U}(t,0)\|\big[\|g(z)(0)-g(\tilde{z})(0)\| \\ &+ \|f(0,g(z)(0)+\phi(0))-f(0,g(\tilde{z})(0)+\phi(0))\|\big] \\ &+ \int_0^t \|\mathcal{U}(t,s)\|\|A_0(s)\|\|f(s,z_s)-f(s,\tilde{z}_s)\|ds \\ &+ \int_0^t \|\mathcal{U}(t,s)\|\|F(s,z_s)-F(s,\tilde{z}_s)\|ds \\ &+ \sum_{0<t_k<t} \|\mathcal{U}(t,t_k)\|\|J_k(t_k,z(t_k))-J_k(t_k,\tilde{z}(t_k))\| \\ &\leq L_f\|z_t-\tilde{z}_\tau(t)\| + M\big[L_g\|z-\tilde{z}\|_{\mathcal{PW}_p} + L_f\|g_\tau(z)(0)-g_\tau(\tilde{z}(0)\|\big] \\ &+ M\|A_0\|L_f\int_0^t \|z_s-\tilde{z}_s\|ds + ML_F\int_0^t \|z_s-\tilde{z}_s\|ds \\ &+ M\sum_{0<t_k<t} d_k\|z(t_k)-\tilde{z}(t_k)\| \\ &\leq \left(L_f + M\left[L_g + L_fL_g + \|A_0\|L_f\tau + L_F\tau + \sum_{k=1}^{p} d_k\right]\right)\|z-\tilde{z}\|_p.\end{aligned}$$

Thus,

$$\|\mathcal{T}z-\mathcal{T}\tilde{z}\|_p \leq \left(L_f + M\left[L_g + L_fL_g + \|A_0\|L_f\tau + L_F\tau + \sum_{k=1}^{p} d_k\right]\right)\|z-\tilde{z}\|_p,$$

so, the operator $\mathcal{T}$ satisfies all the assumptions of the Banach Contraction Mapping Theorem, and therefore $\mathcal{T}$ has only one fixed point in the space $\mathcal{PW}_r$, which is the solution of problem (4.1). This completes the proof. □

5. Example

In this section, we consider an example of semilinear neutral differential equations with impulses, delay and nonlocal conditions such that Theorem 4.1 can be applied. Let us consider the following system

$$\begin{cases} \frac{d}{dt}\left[z(t) - \left(1 + \frac{\tan z(t-2)}{8(t+10)^2}\right)\right] = z(t) + e^{-\frac{z(t-2)}{10(t+5)^3}}, \quad t \in [0,\tau) \\ z(s) = \left(1 + \frac{\sin z}{30^2}\right)(s) + \phi(s), \quad s \in [-2,0] \\ z(t_k^+) = z(t_k^-) + 1 + \frac{\cos\left(z(t_k^-)\right)}{4(t_k+8)^4}, \quad k = 1,2. \end{cases} \tag{5.1}$$

Here $t_1 = \frac{5}{2}$, $t_2 = \frac{9}{2}$ and $\tau = 5$. In this example, the terms related to system (4.1) are given by: $f(t,z) = 1 + \frac{\tan(z)}{8(t+10)^2}$, $F(t,z) = e^{-\frac{z}{10(t+5)^3}}$, $g(z) = 1 + \frac{\sin(z)}{30^2}$, $J_k(t,z) = 1 + \frac{\cos(z)}{4(t+8)^4}$ and $A_0(t) = 1$. Then we have,

$$\begin{array}{rcl}|f(t,z)-f(t,\tilde{z})| & = & \frac{1}{8(t+10)^2}|\tan(z)-\tan(\tilde{z})| \leq \frac{1}{8\cdot 10^2}|z-\tilde{z}|_r,\\ |F(t,z)-F(t,\tilde{z})| & = & \left|e^{-\frac{z}{10(t+5)^3}}-e^{-\frac{\tilde{z}}{10(t+5)^3}}\right| \leq \frac{1}{10\cdot 5^3}|z-\tilde{z}|_r,\\ |J_k(t,z)-J_k(t,\tilde{z})| & = & \frac{1}{4(t+8)^4}|\cos(z)-\cos(\tilde{z})| \leq \frac{1}{4\cdot 8^4}|z-\tilde{z}|_r,\\ |g(z)-g(\tilde{z})|_r & = & \frac{1}{30^2}|\sin(z)-\sin(\tilde{z})|_{\mathcal{PW}_r} \leq \frac{1}{30^2}|z-\tilde{z}|_r,\end{array}$$

and

$$L_f + M[L_g + L_f L_g + |A_0|L_f\tau + L_F\tau + d_1 + d_2] \leq 0.63.$$

Hence, the conditions (L1)-(L4) are satisfied. Consequently, Theorem 4.1 ensures the existence of solutions for problem (5.1).

6. Final Remark

In this paper, first of all, we have proved the existence, uniqueness, and the globally defined solutions of a semilinear neutral differential equation with impulses and nonlocal conditions assuming that the nonlinear terms are locally Lipschitz. After that, we assume that the nonlinear functions that involve system (4.1) are globally Lipschitz, which allows us to prove the existence and uniqueness of solutions by applying Banach Fixed Point Theorem. Finally, we believe that this work can be extended to infinite dimension systems in Hilbert spaces, where the operator A_0 is no longer a matrix, instead, it will be the infinitesimal generator of a strongly continuous compact semigroup, and $-A_0$ a sectorial operator. In this way, the fractional powered spaces can be defined, allowing us to admit nonlinear terms involving spatial derivatives, like in the following neutral partial differential equations of Burges equation type:

$$\begin{cases} \dfrac{\partial}{\partial t}\left[z(t,x)+\displaystyle\int_0^t\int_0^\pi b(\theta-t,y,x)z(\theta,y)dyd\theta\right] = \nu z_{xx}(t,x) - z(t-r)z_x(t-r) \\ +f(t,z(t-r,x)), \quad t \neq t_k, \\ z(t,0)=z(t,1)=0, \quad t\in[0,\tau] \\ z(\theta,x)+h(z(\tau_1+\theta,x),\ldots,z(\tau_q+\theta,x))=\eta(\theta,x), \quad x\in[0,1], \\ z(t_k^+,x)=z(t_k^-,x)+J_k(z(t_k,x)), \quad x\in\Omega, \quad k=1,2,3,\ldots,p, \end{cases} \tag{6.1}$$

where $\eta \in \mathcal{PW}_{1/2}(-r,0;H_0^1) = \mathcal{PW}_{1/2}(-r,0;Z^{1/2})$, with $Z = L_2[0,1]$, $Z^{1/2} = D((-\Delta)^{1/2})$ and the functions f, J_k, h are locally Lipschitz.

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