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De la Vallée Poussin Summability, the Combinatorial Sum $\sum_{k=n}^{2n-1}\binom{2k}{k}$ and the de la Vallée Poussin Means Expansion

Ziad S. Ali

Abstract: In this paper we apply the de la Vallée Poussin sum to a combinatorial Chebyshev sum by Ziad S. Ali in [1]. One outcome of this consideration is the main lemma proving the following combinatorial identity: with $Re(z)$ standing for the real part of z we have

$$\sum_{k=n}^{2n-1}\binom{2k}{k} = Re\left(\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4)\right.$$
$$\left. - \binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)\right).$$

Our main lemma will indicate in its proof that the hypergeometric factors

$${}_2F_1(1,1/2+n;1+n;4), \quad \text{and} \quad {}_2F_1(1,1/2+2n;1+2n;4)$$

are complex, each having a real and imaginary part.

As we apply the de la Vallée Poussin sum to the combinatorial Chebyshev sum generated in the Key lemma by Ziad S. Ali in [1], we see in the proof of the main lemma the extreme importance of the use of the main properties of the gamma function. This represents a second important consideration.

A third new outcome are two interesting identities of the hypergeometric type with their new Meijer G function analogues. A fourth outcome is that by the use of the Cauchy integral formula for the derivatives we are able to give a different meaning to the sum:

$$\sum_{k=n}^{2n-1}\binom{2k}{k}.$$

A fifth outcome is that by the use of the Gauss-Kummer formula we are able to make better sense of the expressions

$$\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4), \quad \text{and} \quad \binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)$$

by making use of the series definition of the hypergeometric function. As we continue we notice a new close relation of the Key lemma, and the de la Vallée Poussin means. With this close relation we were able to talk about the de la Vallée Poussin summability of the two infinite series $\sum_{n=0}^{\infty}\cos n\theta$, and $\sum_{n=0}^{\infty}(-1)^n\cos n\theta$.

Furthermore the application of the de la Vallée Poussin sum to the Key lemma has created two new expansions representing the following functions:

$$\frac{2^{(n-1)}(1+x)^n(-1+2^n(1+x)^n)}{n(2x+1)}, \quad \text{where} \quad x=\cos\theta,$$

and

$$\frac{-2^{(n-1)}(-1+2^n(1-x)^n)(1-x)^n}{n(2x-1)}, \quad \text{where} \quad x=\cos\theta$$

in terms of the de la Vallée Poussin means of the two infinite series

$$\sum_{n=0}^{\infty}\cos n\theta\ ,$$

and

$$\sum_{n=0}^{\infty}(-1)^n\cos n\theta\ .$$

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1. Introduction

The Gauss' Hypergeometric function is given by:

$${}_2F_1(a,b;c;z)=1+\frac{ab}{1\cdot c}z+\frac{a(a+1)b(b+1)}{1\cdot 2\cdot c(c+1)}z^2+\ldots=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n\,!},$$

where the above series converges for $|z|<1$ and $(a)_n$ is is the Pochhammer symbol defined by:

$$(a)_0=0, \quad \text{and} \quad (a)_n=a(a+1)(a+2)\ldots(a+n-1)\ .$$

We further note that:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},$$

where the symbol Γ refers to the gamma function.

Now we like to bring up two definitions related to de la Vallée Poussin; one is related to the de la Vallée Poussin means, and the other is related to the de la Vallée Poussin sum. We have: the de la Vallée Poussin means of the infinite series

$$\sum_{n=0}^{\infty} a_n$$

are defined by (see [3]):

$$V(n, a_n) = \sum_{j=0}^{n} \frac{(n!)^2}{(n-j)!(n+j)!} a_j \ .$$

Let T_i be the Chebyshev polynomials. Then Charles Jean de la Vallée-Poussin defines (see [2]) the de la Vallée Poussin sum SV_n as follows:

$$SV_n = \frac{S_n + S_{n+1} + \ldots + S_{2n-1}}{n},$$

where

$$S_n = \frac{1}{2} c_0(f) + \sum_{j=1}^{n} c_j(f) T_j \ .$$

We shall refer to the S_n as the Chebyshev sum or the Chebyshev expansion of f. The very important properties of the gamma functions that were used in this work, and helped immensely in the proof of the main lemma are:

$$\Gamma(z+1) = z\Gamma(z), \quad \text{and} \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \ .$$

Let f by holomorphic in an open subset U of the complex plane $\mathbb{C}$, and let $|z - z_0| \leq r$, be contained completely in U. Now let a be any point interior to $|z - z_0| \leq r$. Then the Cauchy integral formula for the derivative says that the n-th derivative of f at a is given by:

$$f^n(a) = \frac{n!}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{(z-a)^{n+1}} dz \ .$$

The Meijer G function is defined by

$$G_{p,q}^{m,n} \left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s ds \ .$$

The L in the integral represents the path of integration. We may choose L to run from $-i\infty$ to $+i\infty$ such that all poles of $\Gamma(b_j - s), j = 1, 2, \ldots, m$, are on the right of the path, while all poles of $\Gamma(1 - a_k + s), k = 1, 2, \ldots, n$, are on the left.

The Meijer G function, and the Hypergeometric function ${}_pF_q$ are related by:

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right) = \frac{\prod_{k=1}^{q} \Gamma(b_k)}{\prod_{k=1}^{p} \Gamma(a_k)} \, G^{1,p}_{p,q-1}\left(\begin{matrix} 1 - a_1, \ldots, 1 - a_p \\ 0, 1 - b_1, \ldots 1 - b_q \end{matrix}; -z\right) . $$

2. The close relation between the Key lemma and the de la Vallée Poussin means

In [1] we have the following Key lemma:

Lemma 2.1. *For $1 \le r \le n$, and θ real we have:*

(i) $$\sum_{r=1}^{n} \binom{2n}{n-r} \left(\cos r\theta + (-1)^{r+1}\right) = 2^{n-1}(1 + \cos\theta)^n.$$

(ii) $$\sum_{r=1}^{n} (-1)^r \binom{2n}{n-r} \cos r\theta + \frac{1}{2}\binom{2n}{n} = 2^{n-1}(1 - \cos\theta)^n .$$

Before we move to the main lemma and its proof we state two theorems which are direct consequences of the Key lemma. We like to indicate before we state them that they are related to the de la Vallée Poussin means of the the following two infinite series:

$$\sum_{n=0}^{\infty} \cos n\theta, \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n \cos n\theta .$$

Clearly from the Key lemma we have :

$$\sum_{r=0}^{n} \binom{2n}{n-r} \cos r\theta - \frac{1}{2}\binom{2n}{n} = 2^{n-1}(1 + \cos\theta)^n ,$$

$$\sum_{r=0}^{n} (-1)^r \binom{2n}{n-r} \cos r\theta - \frac{1}{2}\binom{2n}{n} = 2^{n-1}(1 - \cos\theta)^n .$$

Accordingly with

$$V(n, \cos n\theta)$$

being the de la Vallée Poussin means of

$$\sum_{n=0}^{\infty} \cos n\theta,$$

and

$$V(n, (-1)^n \cos n\theta)$$

being the de la Vallée Poussin means of

$$\sum_{n=0}^{\infty} (-1)^n \cos n\theta \ .$$

We have:

$$V(n, \cos n\theta) = \frac{2^{n-1}(1+\cos\theta)^n}{\binom{2n}{n}} + \frac{1}{2} \ ,$$

and

$$V(n, (-1)^n \cos n\theta) = \frac{2^{n-1}(1-\cos\theta)^n}{\binom{2n}{n}} + \frac{1}{2} \ .$$

Now it can be shown that for each real $\theta \neq 2k\pi$, where k is an integer the following limit:

$$\lim_{n\to\infty} \frac{2^{n-1}(1+\cos\theta)^n}{\binom{2n}{n}} = 0 \ , \quad \text{and}$$

for each real $\theta \neq k\pi$,where k is an odd integer the following limit

$$\lim_{n\to\infty} \frac{2^{n-1}(1-\cos\theta)^n}{\binom{2n}{n}} = 0 \ .$$

One way of showing the above limits is the use of the Stirling's formula for the gamma function which says:

$$\Gamma(x) \sim e^{-x}\sqrt{(2\pi)}x^{-1/2+x} \quad \text{as} \quad x \to \ \infty \ .$$

Accordingly we have the following two theorems:

Theorem 2.2. *For each real $\theta \neq 2k\pi$, where k is an integer the infinite series*

$$\sum_{n=0}^{\infty} \cos n\theta$$

is summable in the sense of de la Vallée Poussin to $\frac{1}{2}$.

Theorem 2.3. *For each real $\theta \neq k\pi$, where k is an odd integer the infinite series*

$$\sum_{n=0}^{\infty} (-1)^n \cos n\theta$$

is summable in the sense of de la Vallée Poussin to $\frac{1}{2}$.

3. The main lemma

There are two versions of the main lemma. We give now one version, and we will provide the other one at the end of the current section.

Lemma 3.1. *Let ${}_2F_1(a,b;c;z)$ be the Gauss' Hypergeometric function, then we have:*

$$\sum_{k=n}^{k=2n-1}\binom{2k}{k}=\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4)-\binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)\ .$$

Before we prove the above lemma we like to indicate that by the Gauss-Kummer formula it is easily seen that the hypergeometric functions ${}_2F_1(1,1/2+n;1+n;4)$, and ${}_2F_1(1,1/2+2n;1+2n;4)$ are both discrete complex valued functions, and they are interesting as for exmple the left hand side of the identity given above is a natural number, which leads us to say that the imaginary part of the right hand side of the identity above is actually zero. Another interesting view is that the second term in the above lemma is obtained by replacing n in the first term by $2n$. The further interesting ideas involved are coming as we continue to prove the above lemma, and continue further.

Proof of the case $n=1$ of Lemma 3.1. By induction on n. We note first that for $n=1$ we have:

$$2=2\ {}_2F_1(1,3/2;2;4)-6\ {}_2F_1(1,5/2;3;4)\ .$$

Now one way to evaluate the above expression is by using an already known technique, where the numbers 1, 3/2, 2, 4 inside the hypergeometric function ${}_2F_1$ for example in this case are fed into their proper slot of a computer program or a plug in algorithm which evaluates the hypergeometric functions of the type given above to get the complex number representing ${}_2F_1(1,3/2;2;4)$. Similarly the numbers 1, 5/2, 3, 4 are fed into each slot in the computer program to get the complex number representing ${}_2F_1(1,5/2;3;4)$. For example we see by using this method we have:

$$2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{6}\right)-6\left(-\frac{1}{2}-i\frac{\sqrt{3}}{18}\right),$$

and the identity given in the lemma is true for $n=1$.

One way to prove the case for $n=1$ without the use of a computer is by the use of the **Gauss-Kummer formula**, which states: for $(a-b)$ not an integer, and z not in the unit interval $(0,1)$ we have

$${}_2F_1(a,b;c;z)=\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}\ {}_2F_1(a,a-c+1;a-b+1;1/z)$$

$$+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}\ {}_2F_1(b,b-c+1;-a+b+1;1/z)\ .$$

We will now do basic calculations for the proof of Corollary 3.2, and Corollary 3.3 coming up. Accordingly we have:

$$ {}_2F_1(1,3/2;2;4) = \frac{\Gamma(1/2)\Gamma(2)}{\Gamma(3/2)\Gamma(1)}(-4)^{-1}\ {}_2F_1(1,0;1/2;1/4) $$

$$ + \frac{\Gamma(-1/2)\Gamma(2)}{\Gamma(1)\Gamma(1/2)}(-4)^{-3/2}\ {}_2F_1(3/2,1/2;3/2;1/4)\ . $$

Now since

$$ \Gamma(1/2) = \sqrt{\pi}, \Gamma(3/2) = \frac{\sqrt{\pi}}{2}, \Gamma(-1/2) = -2\sqrt{\pi} $$

we have:

$$ {}_2F_1(1,3/2;2;4) = \frac{\Gamma(1/2)\Gamma(2)}{\Gamma(3/2)\Gamma(1)}(-4)^{-1}\ {}_2F_1(1,0;1/2;1/4) = \frac{-1}{2}\ . $$

Note that

$$ \sum_{n=0}^{\infty} \frac{(0)_n(1)_n}{(1/2)_n 4^n n!} = 1\ . $$

Now since

$$ {}_2F_1(3/2,1/2;3/2;1/4) = \frac{1}{(1-(1/4))^{1/2}} = \frac{2\sqrt{3}}{3}, $$

and $(-4)^{-3/2} = \frac{i}{8}$, we have:

$$ \frac{\Gamma(-1/2)\Gamma(2)}{\Gamma(1)\Gamma(1/2)}(-4)^{-3/2}\ {}_2F_1(3/2,1/2;3/2;1/4) = -i\frac{\sqrt{3}}{6}\ . $$

We like to note now that by using the binomial expansion formula

$$ (x+a)^{\gamma} = \sum_{k=0}^{\infty} \binom{\gamma}{k} x^k a^{\gamma-k}, $$

we have for $|z| < 1$:

$$ (-z+1)^{-1/2} = \sum_{k=0}^{\infty} (-1)^k z^k \binom{\frac{-1}{2}}{k} = {}_2F_1(3/2,1/2;3/2;1/4)\ . $$

Similarly we have:

$$ {}_2F_1(1,5/2;3;4) = \frac{\Gamma(3/2)\Gamma(3)}{\Gamma(5/2)\Gamma(2)}(-4)^{-1}\ {}_2F_1(1,-1;-1/2;1/4) $$

$$ + \frac{\Gamma(-3/2)\Gamma(3)}{\Gamma(1)\Gamma(1/2)}(-4)^{-5/2}\ {}_2F_1(5/2,1/2;5/2;1/4)\ , $$

which simplifies to:

$$-\frac{4}{3}\frac{1}{4}\frac{3}{2} - \frac{i}{32}\frac{8}{3}\frac{2\sqrt{3}}{3} = \frac{-1}{2} - \frac{i\sqrt{3}}{18} \ .$$

Note that

$$_2F_1(1,-1;-1/2;1/4) = \sum_{k=0}^{\infty} \frac{(1)_k(-1)_k}{(-1/2)_k 4^k k!} = \frac{3}{2} \ .$$

For $k \geq 2$ the term $(-1+1)$ is present in each product of $(-1)_k$: accordingly we are adding the first two terms only; furthermore

$$_2F_1(5/2,1/2;5/2;1/4) = \frac{2\sqrt{3}}{3} \ .$$

This is the case where a = c = 5/2, and in this case the sum is

$$\frac{1}{\sqrt{1-1/4}} = \frac{2\sqrt{3}}{2} \ .$$

This completes the induction proof for the case $n = 1$ without the use of a computer. □

We can clearly see from above that $_2F_1(1,3/2;2;4)$, and $_2F_1(1,5/2;3;4)$ are complex numbers, while $_2F_1(1,0;1/2;1/4)$, $_2F_1(3/2,1/2;3/2;1/4)$, $_2F_1(1,-1;-1/2;1/4)$, and $_2F_1(5/2,1/2;5/2;1/4)$ are real numbers.

From above we have the following Corollaries resulting from the induction proof when n = 1, and in relation to the real part, and the imaginary part of $_2F_1(1,3/2;2;4)$, and $_2F_1(1,5/2;3;4)$.

Corollary 3.2. *Let i be the imaginary unit, let $Re(z)$, and $Im(z)$ respectively denote the real part, and the imaginary part of z. We have:*

$$Re\ _2F_1(1,3/2;2;4) = \frac{\Gamma(1/2)\Gamma(2)}{\Gamma(3/2)\Gamma(1)}(-4)^{-1}\ _2F_1(1,0;1/2;1/4) = -\frac{1}{2} \ ,$$

and

$$i\ Im\ _2F_1(1,3/2;2;4) = \frac{\Gamma(-1/2)\Gamma(2)}{\Gamma(1)\Gamma(1/2)}(-4)^{-3/2}\ _2F_1(3/2,1/2;3/2;1/4) = -i\frac{\sqrt{3}}{6} \ .$$

Corollary 3.3. *Let i be the imaginary unit, let $Re(z)$, and $Im(z)$ respectively denote the real part, and the imaginary part of z. We have:*

$$Re\ {}_2F_1(1,5/2;3;4) = \frac{\Gamma(3/2)\Gamma(3)}{\Gamma(5/2)\Gamma(2)}(-4)^{-1}\ {}_2F_1(1,-1;-1/2;1/4) = -\frac{1}{2},$$

and

$$i\ Im\ {}_2F_1(1,5/2;3;4) = \frac{\Gamma(-3/2)\Gamma(3)}{\Gamma(1)\Gamma(1/2)}(-4)^{-5/2}\ {}_2F_1(5/2,1/2;5/2;1/4) = -i\frac{2\sqrt{3}}{18}\ .$$

Accordingly we have the following general lemmas:

Lemma 3.4. *Let i be the imaginary unit, let $Re(z)$, and $Im(z)$ respectively denote the real part, and the imaginary part of z. We have:*

$$Re\ {}_2F_1(1,1/2+n;1+n;4) = \frac{\Gamma(n-1/2)\Gamma(n+1)}{\Gamma(n+1/2)\Gamma(n)}(-4)^{-1}{}_2F_1(1,1-n;3/2-n;1/4)$$

$$i\ Im\ {}_2F_1(1,1/2+n;1+n;4) = \frac{\Gamma(1/2-n)\Gamma(n+1)}{\Gamma(1)\Gamma(1/2)}(-4)^{-1/2-n}$$
$$\times\ {}_2F_1(1/2+n,1/2;1/2+n;1/4)\ .$$

The proof of the above lemma follows by using the Gauss-Kummer formula, and by the presence of $(-4)^{-1/2-n}$ in the second equation.

Lemma 3.5. *Let i be the imaginary unit, let $Re(z)$, and $Im(z)$ respectively denote the real part, and the imaginary part of z. We have:*

$$Re\ {}_2F_1(1,1/2+2n;1+2n;4) = \frac{\Gamma(2n-1/2)\Gamma(2n+1)}{\Gamma(2n+1/2)\Gamma(2n)}(-4)^{-1}{}_2F_1(1,1-2n;3/2-2n;1/4)$$

$$i\ Im\ {}_2F_1(1,1/2+2n;1+2n;4) = \frac{\Gamma(1/2-2n)\Gamma(2n+1)}{\Gamma(1)\Gamma(1/2)}(-4)^{-1/2-2n}$$
$$\times\ {}_2F_1(1/2+2n,1/2;1/2+2n;1/4)\ .$$

The proof of the above lemma follows by using the Gauss-Kummer formula, and by the presence of $(-4)^{-1/2-2n}$ in the second equation.

Now we have the following lemma showing that the imaginary part of the right hand side of the main lemma above equals to zero, i.e.:

Lemma 3.6. *Let i be the imaginary unit, and let $Im(z)$ denote the imaginary part of z. We have:*

$$Im\left(\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4)-\binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)\right)=0\ .$$

Proof of Lemma 3.6.

$$\binom{2n}{n}\left(\frac{\Gamma(1/2-n)\Gamma(n+1)}{\Gamma(1)\Gamma(1/2)}(-4)^{-1/2-n}{}_2F_1(1/2+n,1/2;1/2+n;1/4)\right)$$

$$-\binom{4n}{2n}\left(\frac{\Gamma(1/2-2n)\Gamma(2n+1)}{\Gamma(1)\Gamma(1/2)}(-4)^{-1/2-2n}{}_2F_1(1/2+2n,1/2;1/2+2n;1/4)\right)=0\,,$$

clearly

$${}_2F_1(1/2+n,1/2;1/2+n;1/4))={}_2F_1(1/2+2n,1/2;1/2+2n;1/4))=\frac{1}{\sqrt{1-1/4}}=\frac{2\sqrt{3}}{3}\ .$$

Now by noting that

$$\binom{2n}{n}=\frac{2^{2n}\Gamma(n+1/2)}{\sqrt{\pi}\Gamma(n+1)}\quad\text{and}\quad\binom{4n}{2n}=\frac{2^{4n}\Gamma(2n+1/2)}{\sqrt{\pi}\Gamma(2n+1)},$$

what we do now is to substitute the just above identities for $\binom{2n}{n}$, and $\binom{4n}{2n}$, in the above equation whose right hand side is zero; then we use the complement formula of the gamma function which is:

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}$$

to handle both of the following products: $\Gamma(1/2+n)\Gamma(1/2-n)$ and $\Gamma(1/2+2n)\Gamma(1/2-2n)$.

Finally by writing

$$(-4)^{-1/2-jn}=(2^2e^{i\pi})^{-1/2-jn},\quad\text{with}\quad j=1,2\ .$$

We can easily show that the stated lemma above is correct, and that the imaginary part of the right hand side of the main lemma is identically equal to zero. This completes the proof of the above lemma. □

Proof of the induction step. Since the imaginary part of the right hand side of the main lemma given above equals zero. We have:

$$\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4)-\binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)$$
$$=\binom{2n}{n}\frac{\Gamma(n-1/2)\Gamma(n+1)}{\Gamma(n+1/2)\Gamma(n)}(-4)^{-1}{}_2F_1(1,1-n;3/2-n;1/4)$$
$$-\binom{4n}{2n}\frac{\Gamma(2n-1/2)\Gamma(2n+1)}{\Gamma(2n+1/2)\Gamma(2n)}(-4)^{-1}{}_2F_1(1,1-2n;3/2-2n;1/4)\ .$$

Now assume that the identity we are to prove is true for n, we need to show that it is true for $n+1$; i.e. we need to show that:

$$\sum_{k=n+1}^{k=2n+1}\binom{2k}{k}=\binom{2n+2}{n+1}{}_2F_1(1,3/2+n;2+n;4)-\binom{4n+4}{2n+2}{}_2F_1(1,5/2+2n;3+2n;4)$$

equivalently we need to show that:

$$\sum_{k=n+1}^{k=2n+1}\binom{2k}{k}=\binom{2n+2}{n+1}\frac{\Gamma(n+1/2)\Gamma(n+2)}{\Gamma(n+3/2)\Gamma(n+1)}(-4)^{-1}{}_2F_1(1,-n;1/2-n;1/4)$$
$$-\binom{4n+4}{2n+2}\frac{\Gamma(2n+3/2)\Gamma(2n+3)}{\Gamma(2n+5/2)\Gamma(2n+2)}(-4)^{-1}{}_2F_1(1,-1-2n;-1/2-2n;1/4)$$

equivalently again we need to show that:

$$\binom{2n+2}{n+1}\frac{\Gamma(n+1/2)\Gamma(n+2)}{\Gamma(n+3/2)\Gamma(n+1)}(-4)^{-1}{}_2F_1(1,-n;1/2-n;1/4)$$
$$-\binom{4n+4}{2n+2}\frac{\Gamma(2n+3/2)\Gamma(2n+3)}{\Gamma(2n+5/2)\Gamma(2n+2)}(-4)^{-1}{}_2F_1(1,-1-2n;-1/2-2n;1/4)$$
$$=\binom{2n}{n}\frac{\Gamma(n-1/2)\Gamma(n+1)}{\Gamma(n+1/2)\Gamma(n)}(-4)^{-1}{}_2F_1(1,1-n;3/2-n;1/4)$$
$$-\binom{4n}{2n}\frac{\Gamma(2n-1/2)\Gamma(2n+1)}{\Gamma(2n+1/2)\Gamma(2n)}(-4)^{-1}{}_2F_1(1,1-2n;3/2-2n;1/4)$$
$$-\binom{2n}{n}+\binom{4n}{2n}+\binom{4n+2}{2n+1}\ .$$

The above induction step can be shown by noting the following two new hypergeometric identities:

$${}_2F_1(1,-n;1/2-n;1/4)=\frac{-n}{4(\frac{1}{2}-n)}{}_2F_1(1,1-n;3/2-n;1/4)+1\ ,$$

$$ {}_2F_1(1,1-2n;3/2-2n;1/4) = \binom{8n+2}{2n+1}\left(\frac{4n-1}{n}\right) $$
$$ \times\ {}_2F_1(1,-1-2n;-(1/2)-2n;1/4) - \binom{10n+3}{2n+1}\left(\frac{4n-1}{n}\right)\ . $$

The proof of the first hypergeometric identity above is simple, and the proof of the second hypergeometric identity above is one where one really appreciates the simple law of cancellation. □

Based from above, we provide the second version of the main lemma:

Lemma 3.7. *Let $Re(z)$ be the real part of z, then we have:*

$$ \sum_{k=n}^{2n-1}\binom{2k}{k} = Re\left(\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4) - \binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)\right)\ . $$

An interesting corollary is the following:

Corollary 3.8. *Let*

$$ f(n) = \frac{2^{2n}\Gamma(n+1/2)\Gamma(1/2-n)(-4)^{-1/2-n}}{\sqrt{\pi}}, \quad then \quad f(n) = f(2n)\ . $$

4. Presentation of the two new hypergeometric identities above in terms of the Meijer G function

Since the following identities relating the hypergeometric function ${}_2F_1$, and the Meijer $G^{1,2}_{2,2}$ function, hold:

$$ {}_2F_1(1,-n;1/2-n;1/4) = \frac{\Gamma(1/2-n)}{\Gamma(-n)}\ G^{1,2}_{2,2}\left(\begin{matrix}0 & (1+n)\\ 0 & (1/2+n)\end{matrix}; -1/4\right)\ , $$

$$ {}_2F_1(1,1-n;3/2-n;1/4) = \frac{\Gamma(3/2-n)}{\Gamma(1-n)}\ G^{1,2}_{2,2}\left(\begin{matrix}0 & n\\ 0 & (-1/2+n)\end{matrix}; -1/4\right)\ , $$

$$ {}_2F_1(1,1-2n;3/2-2n;1/4) = \frac{\Gamma(3/2-n)}{\Gamma(1-2n)}\ G^{1,2}_{2,2}\left(\begin{matrix}0 & 2n\\ 0 & (-1/2+2n)\end{matrix}; -1/4\right)\ , $$

$$ {}_2F_1(1,-1-2n;-1/2-2n;1/4) = \frac{\Gamma(-1/2-2n)}{\Gamma(-1-2n)}\ G^{1,2}_{2,2}\begin{pmatrix} 0 & (2n+2) \\ 0 & (3/2+2n) \end{pmatrix}; -1/4 \Big) \ , $$

then we have the following two new identities relating the Meijer function $G^{1,2}_{2,2}$:

$$ \frac{\Gamma(1/2-n)}{\Gamma(-n)} G^{1,2}_{2,2}\begin{pmatrix} 0 & (1+n) \\ 0 & (1/2+n) \end{pmatrix}; -1/4 \Big) = \frac{-n}{4(1/2-n)} \frac{\Gamma(3/2-n)}{\Gamma(1-n)} $$

$$ \times\ G^{1,2}_{2,2}\begin{pmatrix} 0 & n \\ 0 & (-1/2+n) \end{pmatrix}; -1/4 \Big) + 1 \ , $$

$$ \frac{\Gamma(3/2-n)}{\Gamma(1-2n)} G^{1,2}_{2,2}\begin{pmatrix} 0 & 2n \\ 0 & (-1/2+2n) \end{pmatrix}; -1/4 \Big) = \left(\frac{8n+2}{2n+1}\right)\left(\frac{4n-1}{n}\right)\frac{\Gamma(-1/2-2n)}{\Gamma(-1-2n)} $$

$$ \times\ G^{1,2}_{2,2}\left(0 \quad (2n+2)0 \quad (3/2+2n); -1/4\right) - \left(\frac{10n+3}{2n+1}\right)\left(\frac{4n-1}{n}\right) \ . $$

5. A different meaning of $\sum_{k=n}^{2n-1}\binom{2k}{k}$

Let $r > 0$ be given:

$$ \binom{m}{r} = \frac{1}{2\pi i}\int_{|z|=r} \frac{(1+z)^m}{z^{r+1}} dz \ . $$

Accordingly:

$$ \binom{2k}{k} = \frac{1}{2\pi i}\int_{|z|=r} \frac{(1+z)^{2k}}{z^{k+1}} dz \ . $$

Hence

$$ \sum_{k=n}^{2n-1} \frac{1}{2\pi i}\int_{|z|=r} \frac{(1+z)^{2k}}{(z-0)^{k+1}} dz = \binom{2n}{n} {}_2F_1(1, 1/2+n; 1+n; 4) $$

$$ - \binom{4n}{2n} {}_2F_1(1, 1/2+2n; 1+2n; 4) \ . $$

Accordingly by the Cauchy integral formula for derivatives we have the following new, and different meaning of $\sum_{k=n}^{2n-1}\binom{2k}{k}$:

$$ \sum_{k=n}^{2n-1}\binom{2k}{k} = \sum_{k=n}^{2n-1} \frac{1}{k!} \frac{\mathrm{d}^k\left((1+z)^{2k}\right)}{\mathrm{d}z^k}\bigg|_{z=0} \ . $$

The above formula clearly states the different meaning of the sum $\sum_{k=n}^{2n-1}\binom{2k}{k}$. We remark that

$$ \frac{\mathrm{d}^k\left((1+z)^{2k}\right)}{\mathrm{d}z^k}\bigg|_{z=0} $$

stands for the k^{th} derivative of of $(1+z)^{2k}$ evaluated at $z = 0$.

6. The de la Vallée Poussin sum, and the de la Vallée Poussin means expansion

In this section, we apply the de la Vallée Poussin sum SV_n to the Key lemma parts (i), and (ii) to obtain an implicit, and an explicit de la Vallée Poussin means expansions DVP_i, and DVP_{ii} of two particular functions, which will be defined in the statements of the theorems. We remark that from the Key lemma we can easily see:

$$S_k = \frac{\binom{2k}{k}}{2} + \sum_{j=1}^{k} \binom{2k}{k-j} \cos j\theta \ .$$

Accordingly, we have the following two theorems:

Theorem 6.1. *The function*

$$\frac{2^{(n-1)}(1+x)^n(-1+2^n(1+x)^n)}{n(2x+1)}, \quad \textit{where} \quad x = \cos\theta$$

has an implicit de la Vallée Poussin means expansion DVP_i of the form

$$= \frac{1}{2n}\sum_{k=n}^{2n-1}\binom{2k}{k} + \frac{1}{n}\sum_{k=n}^{2n-1}\left(\sum_{j=1}^{k}\binom{2k}{k-j}T_j(x)\right)$$

and an explicit de la Vallée Poussin expansion DVP_i of the form

$$= \frac{1}{2n}\sum_{k=n}^{2n-1}\binom{2k}{k} + \frac{1}{n}\sum_{k=n}^{2n-1}\binom{2k}{k}V_{k1} \ ,$$

where V_{k1} are the de la Vallée Poussin means of

$$\sum_{k=1}^{\infty} \cos k\theta \ .$$

Theorem 6.2. *The function*

$$\frac{-2^{(n-1)}(-1+2^n(1-x)^n)(1-x)^n}{n(2x-1)}, \quad \textit{where} \quad x = \cos\theta$$

has an implicit de la Vallée Poussin means expansion DVP_{ii} of the form

$$= \frac{1}{2n}\sum_{k=n}^{2n-1}\binom{2k}{k} + \frac{1}{n}\sum_{k=n}^{2n-1}\left(\sum_{j=1}^{k}(-1)^j\binom{2k}{k-j}T_j(x)\right)$$

and an explicit de la Vallée Poussin expansion DVP_{ii} of the form

$$= \quad \frac{1}{2n}\sum_{k=n}^{2n-1}\binom{2k}{k}+\frac{1}{n}\sum_{k=n}^{2n-1}\binom{2k}{k}V_{k2}\ ,$$

where V_{k2} are the de la Vallée Poussin means of

$$\sum_{k=1}^{\infty}(-1)^k\cos k\theta\ .$$

7. Concluding remark

In this paper, we were able to show that

$$\sum_{k=n}^{2n-1}\binom{2k}{k}=Re\left(\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4)-\binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)\right).$$

Moreover, we were able to create two new hypergeometric identities to prove the induction step. After, it was interesting to find their Meijer G function analogue. Further, by the use of the Key lemma and the definition of the de la Vallée Poussin means, we were able to find two new expansions representing the following functions:

$$\frac{2^{(n-1)}(1+x)^n(-1+2^n(1+x)^n)}{n(2x+1)},\quad \text{where}\quad x=\cos\theta$$

and

$$\frac{-2^{(n-1)}(-1+2^n(1-x)^n)(1-x)^n}{n(2x-1)},\quad \text{where}\quad x=\cos\theta.$$

The general form of the expansions can be put into a more familiar form as:

$$\frac{A_0}{2}+\sum_{K=1}^{n}A_{(K,n)}V_{(K,n)},\quad \text{where}$$

$$A_0=\quad \frac{1}{n}\sum_{K=1}^{n}\binom{2K+2n-2}{K+n-1},\quad \text{and}$$

$$A_{(K,n)}=\frac{1}{n}\binom{2K+2n-2}{K+n-1}\ .$$

We like to add that the strong connection of the Key lemma, and the de la Vallée Poussin means has given us two theorems about the de la Vallée Poussin summability of the two infinite series

$$\sum_{n=0}^{\infty}\cos n\theta \qquad \theta\neq 2k\pi \quad k \quad \text{integer},\quad \text{and}$$

$$\sum_{n=0}^{\infty}(-1)^n \cos n\theta \qquad \theta \neq k\pi \quad k \quad \text{odd} \quad \text{integer}.$$

Moreover we note for example that

$$\sum_{n=0}^{\infty}(-1)^n$$

is also de la Vallée Poussin summable to $\frac{1}{2}$ just like it's Cesàro sum, and it's Abel sum.
We remark finaly that:

$$\sum_{K=1}^{n}\binom{2K+2n-2}{K+n-1} = Re\left(\binom{2n}{n}{}_2F_1(1,1/2+n;1+n;4)\right.$$
$$\left. - \binom{4n}{2n}{}_2F_1(1,1/2+2n;1+2n;4)\right).$$

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On Regulated Functions

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Abstract: The main purpose of this review article is to present the concept of a regulated function and to indicate the connection of the class of regulated functions with other significant classes of functions. In particular, we give a characterization of regulated functions in terms of step functions and we show that the linear space of regulated functions forms a Banach space under the classical supremum norm.

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1. Introduction

The paper is dedicated to present some basic facts concerning the so-called regulated functions. The class of those functions is very important in the theory of functions of a real variable and is especially exploited in the description and characterization of a lot of classes of functions of generalized bounded variation (cf. [1]). If is worthwhile mentioning that instead of the term "regulated function" we use also sometimes the term "regular function" [1].

It seems that the concept of a regulated function was introduced by G. Aumann in his monograph [2]. In that monograph we can find the proof of the fact that the space of regulated functions forms a Banach space but the presented proof of this fact seems to be complicated and a bit incomprehensible. The excellent proof of the mentioned fact was given in the famous book of J. Dieudonné [3]. Our presentation of the theory of regulated functions is closely patterned on the mentioned book.
Nevertheless, the proof given in [3] contains a few gaps and errors which will be improved and completed in the present paper.

As we mentioned previously the concept of a regulated function is used in investigations concerning a few classes of functions of generalized bounded variation (cf. [5, 8, 9], for example). A comprehensive presentation of several classes of functions of generalized bounded variation and their connections with the class of regulated functions was presented in the book [1].

Let us also indicate that the class of regulated functions was also used in the study of stochastic integral equations [6]. Some investigations of regulated functions were also conducted in the paper [4] in connection with the description of classes of functions which are relatively compact in the space of regulated functions. On the other hand it seems that the results obtained in that paper are not entirely satisfactory from the view point of possible applications.

The paper has a review character and it can be viewed as an introduction to further study of some problems related to the theory of regulated functions. The details will appear elsewhere.

2. Auxiliary facts

The basic tool used in the paper is the concept of a metric space. Thus, let us denote by (X, d) a metric space. If Y is a nonempty subset of X then it can be regarded as a metric subspace of the space X with the metric induced by d.
For further purposes we will denote by $B(x, r)$ the open ball (in the metric space X) with the center at x and with radius r, respectively.
Throughout the paper we will use the standard concepts and notation of the theory of metric space (cf. [3, 10]). For example, if A is a subset of the metric space X then we denote by $\overline{A}$ its closure. Obviously, if $A = \overline{A}$ then A is called a closed set. If Y is a subspace of the metric space X and A is a subset of Y such that $\overline{A} = Y$, then we say that A is dense in Y. Moreover, in the standard way we define the concept of a relatively compact and compact set in the metric space X [3, 10].

Now, we recall a few classical facts which will be used in our study [3].

Theorem 2.1. *Let (X, d) be a complete metric space. Then any nonempty, closed subset of the space X is a complete subspace of X.*

Theorem 2.2. *Let (X, d) be a metric space and let A be a nonempty subset of X such that A forms a complete subspace of X. Then the set A is closed.*

In what follows we will discuss the concept of an isolated point and an isolated set [11]. Namely, if (X, d) is a metric space and A is a subset of X then a point $x \in A$ is called an isolated point of A if it is not an accumulation point of A i.e., there exists $r > 0$ such that $B(x, r) \cap A = \{x\}$.

A subset A of the metric space X is said to be an isolated set if each point of A is an isolated point of the set A.
We have the following theorem.

Theorem 2.3. *The set of all isolated points of a set A is an isolated set.*

Proof. The proof requires the standard reasoning. Namely, denote by isA the set of all isolated points of the set A. Let $y \in isA$. Then there exists $r > 0$ such that $B(y, r) \cap A = \{y\}$. Hence we infer that the ball $B(y, r)$ does not contain points of the set isA, except the point y. Indeed, it is a simple consequence of the fact that each point of the set isA belongs to the set A. On the other hand the ball $B(y, r)$ contains only one point of the set A, the point y. Thus y is an isolated point of the set isA. □

For our further purposes the next theorem will be essential.

Theorem 2.4. *Let (X, d) be a separable metric space. Then every isolated subset A of the space X is at most countable.*

In order to make the paper self-contained we give the proof of this theorem (cf. [11]).

Proof. Let $W = \{y_1, y_2, \ldots\}$ be an at most countable dense subset of X. For arbitrarily fixed point $x \in A$ denote by r_x a positive number such that $B(x, r_x) \cap A = \{x\}$. Next, for an arbitrary $x \in A$ we will denote by $n(x)$ the least natural number such that $y_{n(x)} \in B(x, \frac{1}{2}r_x)$. In this way we define the function $n : A \to \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers.
We show that this function is an injection. To this end fix arbitrary $x, x' \in A$ and assume that $n(x) = n(x')$. This means that $y_{n(x)} = y_{n(x')}$. Let us put

$$y = y_{n(x)} = y_{n(x')}.$$

Then we have that $y = y_{n(x)} \in B(x, \frac{1}{2}r_x)$ and, similarly $y = y_{n(x')} \in B(x', \frac{1}{2}r_{x'})$. Hence we obtain

$$d(x, y) < \frac{1}{2}r_x, \quad d(x', y) < \frac{1}{2}r_{x'}.$$

Consequently, we get

$$d(x, x') \leqslant d(x, y) + d(y, x') < \frac{1}{2}(r_x + r_{x'}).$$

Suppose that $r_x \leqslant r_{x'}$. Then the above inequality implies that $d(x, x') < r_{x'}$. This allows us to deduce that

$$x \in B(x', r_{x'}) \cap A = \{x'\},$$

which gives that $x = x'$.
In the case when $r_{x'} \leqslant r_x$, the similar reasoning leads to the same conclusion. Finally, we conclude that the function $n = n(x)$ is an injective mapping. This means that the set A has the same cardinality as a certain subset of the set $\mathbb{N}$. The proof is complete. □

From the above theorem we obtain the following useful corollary.

Corollary 2.5. *Let (X, d) be a metric space. If there exists an uncountable subset A of X and a number $\varepsilon > 0$ such that for arbitrary $x, y \in A$, $x \neq y$, we have that $d(x, y) \geqslant \varepsilon$, then the space X is not separable.*

Proof. In view of the assumption we infer that the set A is isolated. Indeed, for an arbitrary $x \in A$ we have

$$B(x, \varepsilon) \cap A = \{x\}.$$

If X would be separable then in view of Theorem 2.4 we have that A is at most countable. The obtained contradiction completes the proof. □

To illustrate the usefulness of Corollary 2.5 let us consider the following example.

Example 2.6. Consider the Banach sequence space l_∞ consisting of all real bounded sequences and normed with help of the supremum norm. Then the set A of all sequences with terms equal 0 or 1 is uncountable. Moreover, for $x, y \in A$, $x \neq y$, we have $d(x,y) = 1$. Hence, in view of Corollary 2.5 we conclude that l_∞ is not separable.

The next theorem will play a crucial role in our considerations.

Theorem 2.7. *Let A, B be nonempty subsets of the metric space X such that $A \subset B$. Assume that for an arbitrary $x \in X$ the following condition is satisfied:*

$$x \in B \text{ if and only if there exists a sequence } (a_n) \subset A \text{ such that } a_n \to x. \tag{D}$$

Then the set B is closed and A is dense in the set B.

Proof. Take $b \in \overline{B}$. Then there exists a sequence $(b_n) \subset B$ such that $b_n \to b$.
Since $b_1 \in B$, we can find a sequence $(a_n^1) \subset A$ such that $a_n^1 \to b_1$.
Further, since $b_2 \in B$, we can find a sequence $(a_n^2) \subset A$ such that $a_n^2 \to b_2$.
Similarly, for an arbitrary natural number k, taking the term $b_k \in B$, we can find a sequence $(a_n^k) \subset A$ such that $a_n^k \to b_k$ as $n \to \infty$.

Now, fix arbitrarily $\varepsilon > 0$ and choose $n_1 \in \mathbb{N}$ such that $d(a_{n_1}^1, b_1) < \frac{\varepsilon}{2}$. Next, we choose $n_2 \in \mathbb{N}$, $n_2 > n_1$, such that $d(a_{n_2}^2, b_2) < \frac{\varepsilon}{2}$ and so on.
Thus we can find a sequence $(a_{n_k}^k) \subset A$ such that $d(a_{n_k}^k, b_k) < \frac{\varepsilon}{2}$ for $k = 1, 2, \ldots$.
Hence we got

$$d(a_{n_k}^k, b) \leqslant d(a_{n_k}^k, b_k) + d(b_k, b) < \varepsilon$$

for k big enough. This implies that $a_{n_k}^k \to b$ if $k \to \infty$. Hence, in view of condition (D) we obtain that $b \in B$. This means that the set B is closed. The conclusion that A is dense in B is obvious. □

In the sequel of the paper we will work in the function space $B([a,b])$ consisting of all real functions defined and bounded on the interval $[a,b]$. Recall that $B([a,b])$ forms a Banach space under the supremum norm which will be denoted by $\|\cdot\|_\infty$ i.e., for $f \in B([a,b])$ we put

$$\|f\|_\infty = \sup\big\{|f(x)| : x \in [a,b]\big\}.$$

The space $B([a,b])$ is not separable. This space is very convenient in numerous investigations conducted in the theory of real functions and functional analysis. Let us pay our attention to two important subspaces of the space $B([a,b])$.

Namely, consider the subset $C([a,b])$ of the space $B([a,b])$ which consists of all functions continuous on the interval $[a,b]$. It is well-known that $C([a,b])$ forms a closed subset of the space $B([a,b])$ under the supremum norm. Thus $C([a,b])$ is the Banach space with the norm $\|\cdot\|_\infty$.

Another important subspace of $B([a,b])$ is formed by the so-called step functions. To describe that space we introduce first the definition of the concept of a step function.

Definition 2.8. A function $f:[a,b]\to\mathbb{R}$ is called a *step function* if there exists a finite sequence $\{x_0,x_1,\ldots,x_n\}\subset[a,b]$ such that $a=x_0<x_1<x_2<\ldots<x_n=b$ and such that the function f is constant on every interval (x_{i-1},x_i) $(i=1,2,\ldots,n)$.

The set of all step functions on the interval $[a,b]$ will be denoted by $S([a,b])$.
Let us observe that $S([a,b])\subset B([a,b])$. Moreover, the set $S([a,b])$ is a linear space over the field of real numbers $\mathbb{R}$ with the usual operations of the addition of functions and the multiplication of a function by a real scalar. To prove this statement let us take arbitrary functions $f,g\in S([a,b])$. Then there exist two finite sets $X=\{x_0,x_1,\ldots,x_n\}$, $Y=\{y_0,y_1,\ldots,y_m\}$ with the property $a=x_0<x_1<\ldots<x_n=b$, $a=y_0<y_1<\ldots<y_m=b$ and such that the function f is constant on each interval (x_{i-1},x_i) for $i=1,2,\ldots,n$ and the function g is constant on each interval (y_{j-1},y_j) for $j=1,2,\ldots,m$. Take the union $X\cup Y$ and arrange the elements of this set into an increasing sequence $Z=\{z_0,z_1,\ldots,z_k\}$ in such a way that if some two elements of the sets X and Y are the same, then we treat them as one point of the set Z. Thus

$$a=z_0<z_1<\ldots<z_k=b.$$

Notice that the functions f and g are constant on each interval (z_{i-1},z_i) for $i=1,2,\ldots,k$. This implies that $f+g$ is also constant on each of the mentioned intervals. Thus $f+g\in S([a,b])$.
Similarly (even in an easier manner) we show that $\alpha f\in S([a,b])$ for any $\alpha\in\mathbb{R}$.
Finally we conclude that $S([a,b])$ is a linear subspace of the space $B([a,b])$. This justifies our earlier assertion.

Now, we show that the space $S([a,b])$ is not complete under the norm induced from the space $B([a,b])$ i.e., under the supremum norm $\|\cdot\|_\infty$.
To this end let us take into account the following example.

Example 2.9. Consider the function $f\in B([0,1])$ defined in the following way

$$f(x)=\begin{cases}\frac{1}{n+1} & \text{for } x\in\left[\frac{1}{n+1},\frac{1}{n}\right),\ n=1,2,\ldots\\ 1 & \text{for } x=1\\ 0 & \text{for } x=0.\end{cases}$$

Observe that f is not a step function since f is not constant on finite family of open subintervals of the interval $[0,1]$. Thus $f\notin S([a,b])$.
Next, let us take the sequence (f_n) of functions defined for each fixed natural number n in the following way

$$f_n(x)=\begin{cases}\frac{1}{k+1} & \text{for } x\in\left[\frac{1}{k+1},\frac{1}{k}\right),\ k=1,2,\ldots,n\\ 1 & \text{for } x=1\\ 0 & \text{for } x\in\left[0,\frac{1}{n+1}\right).\end{cases}$$

It is easily seen that $f_n \in S([0,1])$ for any $n \in \mathbb{N}$. Moreover, for a fixed n we have

$$\|f - f_n\|_\infty = \frac{1}{n+2}.$$

Hence we deduce that the sequence (f_n) converges to the function f in the topology generated by the norm of the space $B([0,1])$. Thus f is a cluster point of the set $S([0,1])$. Since $f \notin S([0,1])$, we infer that the set $S([0,1])$ is not a closed set in the space $B([0,1])$. On the base of Theorem 2.2 this leads to the conclusion that the space $S([0,1])$ is not complete $\big($under the supremum norm $\|\cdot\|_\infty\big)$.

3. Existence of finite limits of a function via the Cauchy condition

It is well-known [3, 7] that the Cauchy condition plays an essential role in mathematical and functional analysis. Obviously, it is very useful in the elementary theory of sequences in metric space, in the theory of series in Banach space and in the theory of real functions [7]. The fundamental importance of the concept of the Cauchy condition relies on the creating of the possibility of the defining of the completeness of a metric space.

In this section we focus on the formulation of the Cauchy condition for real functions, since this condition enables us to obtain handy tools in the theory of regulated functions.

Thus, let us assume that D is a nonempty subset of the set of real numbers $\mathbb{R}$ and let x_0 $(x_0 \in \mathbb{R})$ be an accumulation point of the set D.

Moreover, let $f : D \to \mathbb{R}$ be a given function.

Definition 3.1. We say that the function f satisfies at the point x_0 the Cauchy condition if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{\substack{t,s\in D\\ t\neq x_0\\ s\neq x_0}}{\forall} \big[|t-x_0|<\delta, |s-x_0|<\delta \Rightarrow |f(t)-f(s)|<\varepsilon\big].$$

Similarly, let us assume now that x_0 $(x_0 \in \mathbb{R})$ is a left hand point of accumulation of the set D.

Definition 3.2. We say that the function $f : D \to \mathbb{R}$ satisfies at the point x_0 the left hand Cauchy condition if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{\substack{t,s\in D\\ t<x_0\\ s<x_0}}{\forall} \big[x_0-t<\delta, x_0-s<\delta \Rightarrow |f(t)-f(s)|<\varepsilon\big].$$

In the same way we can formulate the definition of the concept of the right hand Cauchy condition. Namely, assume that x_0 $(x_0 \in \mathbb{R})$ is a right hand point of accumulation of the set D.

Definition 3.3. We say that the function $f : D \to \mathbb{R}$ satisfies at the point x_0 the right hand Cauchy condition if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{\substack{t,s\in D\\ t>x_0\\ s>x_0}}{\forall} \big[t - x_0 < \delta, s - x_0 < \delta \Rightarrow |f(t) - f(s)| < \varepsilon\big].$$

It is well known [7, 11] that the existence of a finite limit of the function $f : D \to \mathbb{R}$ is equivalent to the Cauchy condition. We formulate this result in details.

Theorem 3.4. *Assume that D is a nonempty subset of the set $\mathbb{R}$ and x_0 ($x_0 \in \mathbb{R}$) is an accumulation point of the set D (a left hand accumulation point of D; a right hand accumulation point of D, respectively). Let $f : D \to \mathbb{R}$ be a given function. Then:*

(i) The finite limit $\lim_{x\to x_0} f(x)$ does exist if and only if the function f satisfies the Cauchy condition at the point x_0.

(ii) The finite left hand limit $\lim_{x\to x_0^-} f(x)$ does exist if and only if the function f satisfies the left hand Cauchy condition at the point x_0.

(iii) The finite right hand limit $\lim_{x\to x_0^+} f(x)$ does exist if and only if the function f satisfies the right hand Cauchy condition at the point x_0.

Remark 3.5. Observe that the Cauchy condition for the function $f : D \to \mathbb{R}$ can be also formulated in the case when we assume that $-\infty$ or $+\infty$ is the accumulation point of the set D. Then we can also formulate a suitable version on the existence of finite limits of the function f at $-\infty$ or at $+\infty$, similarly as in the case of Theorem 3.4. We omit details.

4. Regulated functions and their properties

In this section we will discuss the concept of a regulated function. To make our presentation more transparent we restrict ourselves to real functions i.e., to functions with values in the set of real numbers $\mathbb{R}$.
The possible generalization to the case of functions with values in an arbitrary Banach space will be discussed in the next section.

Definition 4.1. A function $f \in B([a, b])$ is called *a regulated function* if it has one-sided limits at every point $x \in (a, b)$ and if it has the right hand limit at $x = a$ and the left hand limit at $x = b$.

Other words, f is regulated on the interval $[a, b]$ if for each $x \in (a, b)$ there exist limits $f(x-)$, $f(x+)$ and there exist limits $f(a+)$, $f(b-)$.

The class of all regulated functions on the interval $[a, b]$ will be denoted by $R([a, b])$.

Observe that the assumption requiring that f is a member of the space $B([a, b])$ implies that the limits indicated in Definition 4.1 are finite.

Now, we show that the concept of a regulated function can be presented in other way if we dispense with the assumption that $f \in B([a, b])$.
Indeed, we have the following theorem.

Theorem 4.2. *Let $f : [a, b] \to \mathbb{R}$ be a function with the property that for each $x_0 \in [a, b]$ there exist finite one-sided limits $f(x_0-)$ and $f(x_0+)$ of the function f at the point x_0 (in the case $x_0 = a$ we assume that there exists the finite one-sided limit $f(a+)$ and similarly, there exists a finite one-sided limit $f(b-)$ in the case $x_0 = b$). Then the function f is bounded on the interval $[a, b]$.*

Proof. Suppose contrary i.e., the function f is not bounded on the interval $[a, b]$. In order to fix our attention let us assume that f is not bounded from above on $[a, b]$. Then there exists a sequence $(x_n) \subset [a, b]$ such that $f(x_n) \geqslant n$ for $n = 1, 2, \dots$. Since the sequence (x_n) is bounded, in view of Bolzano-Weierstrass theorem we infer that there exists a subsequence (x_{k_n}) converging to some point $x_0 \in [a, b]$. Then we have

$$f(x_{k_n}) \geqslant k_n$$

for $n = 1, 2, \dots$. Observe that we can assume that we can select the subsequence (x_{k_n}) in such a way that $x_{k_n} \neq x_0$ for $n = 1, 2, \dots$. If this would be not possible then $x_{k_n} \neq x_0$ only for finite number of terms of the sequence (x_{k_n}) i.e., there would be exist a subsequence (x_{p_n}) of the sequence (x_{k_n}) such that $x_{p_n} = x_0$ for $n = 1, 2, \dots$. Then

$$f(x_{p_n}) = f(x_0) \geqslant p_n$$

for $n = 1, 2, \dots$. Hence, taking into account that $p_n \to \infty$ for $n \to \infty$, we infer that $f(x_0) = \infty$.
The obtained contradiction shows that we can assume that $x_{k_n} \neq x_0$ for $n = 1, 2, \dots$.

Next, let us notice that we can select a subsequence (x_{l_n}) of the sequence (x_{k_n}) such that $x_{l_n} < x_0$ for $n = 1, 2, \dots$ or, we can select a subsequence (x_{q_n}) of (x_{k_n}) such that $x_{q_n} > x_0$ for $n = 1, 2, \dots$. Obviously, both sequences (x_{l_n}) and (x_{q_n}) are converging to x_0. In order to fix our attention let us assume that there exists a subsequence (x_{l_n}) of the sequence (x_{k_n}) such that $x_{l_n} < x_0$ $(n = 1, 2, \dots)$ and $x_{l_n} \to x_0$. Then we have

$$f(x_{l_n}) \geqslant l_n$$

for $n = 1, 2, \dots$. Hence we get that

$$\lim_{n \to \infty} f(x_{l_n}) = \infty.$$

But this contradicts to the assumption that there exists a finite left hand limit $f(x_0-)$. Obviously in the case $x_0 = a$ or $x_0 = b$ the proof is similar. □

The above theorem shows that in the definition of the regulated function f instead of the assumption that $f \in B([a, b])$ we can equivalently assume that the function $f : [a, b] \to \mathbb{R}$ has finite one-sided limits at every point of the interval $[a, b]$.

In what follows let us note that we have the inclusion $R([a,b]) \subset B([a,b])$ which follows immediately from Definition 4.1. Further, observe that $S([a,b]) \subset R([a,b])$ which is a simple consequence of the definitions of a step function and a regulated function. Obviously, the converse inclusion is not valid. Indeed, let us consider again Example 2.9. Then we see that the function f considered in that example is regulated but it is not a step function.

Now, we show that the set $R([a,b])$ forms a linear subspace of the space $B([a,b])$. To prove this fact take arbitrary functions $f, g \in R([a,b])$ and fix a point $x_0 \in (a,b)$. Then there exist the limits $f(x_0-)$, $f(x_0+)$, $g(x_0-)$, $g(x_0+)$. By the standard theorems of mathematical analysis we infer that

$$(f+g)(x_0-) = \lim_{x \to x_0-} \big(f(x) + g(x)\big) = \lim_{x \to x_0-} f(x) + \lim_{x \to x_0-} g(x) = f(x_0-) + g(x_0-).$$

Similarly we show that there exist (finite) limits $(f+g)(x_0+)$ and $(f+g)(a+)$, $(f+g)(b-)$.
Thus $f + g \in R([a,b])$.
The proof showing that $\alpha \cdot f \in R([a,b])$ for $\alpha \in \mathbb{R}$ is also standard.
Thus the set $R([a,b])$ of regulated functions on the interval $[a,b]$ has the algebraic structure of a linear space over the field $\mathbb{R}$.

Moreover, the space $R([a,b])$, as a subspace of the Banach space $B([a,b])$, forms a normed space with respect to the supremum norm $\|\cdot\|_\infty$. However, the proof of the fact, that this norm is complete (i.e., that $R([a,b])$ is a Banach space under the norm $\|\cdot\|_\infty$), is not easy [3].
The key role in the announced proof is played by the following theorem.

Theorem 4.3. *Let $f \in B([a,b])$. Then $f \in R([a,b])$ if and only if there exists a sequence (f_n) of step functions on the interval $[a,b]$ $\big($i.e., $(f_n) \subset S([a,b])\big)$ such that it is uniformly convergent on the interval $[a,b]$ to the function f $\big($equivalently: $\lim_{n\to\infty} \|f_n - f\|_\infty = 0\big)$.*

Proof. At first, we prove the part $\Rightarrow$.
Thus, take a function $f \in R([a,b])$. Fix arbitrarily a natural number n and $x \in [a,b]$. Assume first that $x \in (a,b)$. Then, taking into account Theorem 3.4 we deduce that there exists an interval $\big(y(x), z(x)\big) \subset [a,b]$ such that $x \in \big(y(x), z(x)\big)$ and for arbitrary $t, s \in \big(y(x), x\big)$ we have that $|f(t) - f(s)| \leqslant \frac{1}{n}$ and, for arbitrary $t, s \in \big(x, z(x)\big)$ we have that $|f(t) - f(s)| \leqslant \frac{1}{n}$.
Similarly, taking $x = a$ we can meet an interval $\big[a, z(a)\big) \subset [a,b]$ such that for $t, s \in \big(a, z(a)\big)$ we have $|f(t) - f(s)| \leqslant \frac{1}{n}$. In the same way we find an interval $\big(y(b), b\big] \subset [a,b]$ such that $|f(t) - f(s)| \leqslant \frac{1}{n}$ for $t, s \in \big(y(b), b\big)$.

Further, consider the following family of the intervals:

$$\Big\{\big(y(x), z(x)\big)\Big\}_{x \in (a,b)} \cup \Big\{\big[a, z(a)\big), \big(y(b), b\big]\Big\}.$$

This family forms an open covering of the interval $[a,b]$ (in the topological space formed by the interval $[a,b]$ with the natural metric). Since the space $[a,b]$ is compact we can select a finite family of intervals

$$\Big\{\big(y(x_i), z(x_i)\big)\Big\}_{1\leqslant i\leqslant m} \cup \Big\{\big[a, z(a)\big), \big(y(b), b\big]\Big\},$$

which is a subcovering of the interval $[a, b]$.

Next, let $(c_j)_{0\leqslant j\leqslant k}$ be a strictly increasing finite sequence formed by the numbers $a, b, z(a), y(b), x_i, y(x_i), z(x_i)$ $(i = 1, 2, \ldots, m)$. Obviously $c_0 = a$, $c_k = b$.
Observe that each of the intervals $(c_0, c_1), (c_1, c_2), \ldots, (c_{k-1}, c_k)$ is located in a certain of the selected intervals which form a finite covering of $[a, b]$. On the other hand excluding the first and the last interval, any of such an interval is contained either in an interval $\big(y(x_i), x_i\big)$ or in an interval $\big(x_i, z(x_i)\big)$. Similar assertion holds for the intervals $[c_0, c_1), (c_{k-1}, c_k]$. Thus, for arbitrary $t, s \in (c_{j-1}, c_j)$ $(j = 1, 2, \ldots, k)$ we have that $|f(t) - f(s)| \leqslant \frac{1}{n}$.

Now, for an arbitrarily fixed $j \in \{1, 2, \ldots, k\}$, let us define a function f_n to be equal to the value of the function f at an arbitrary point in the interval (c_{j-1}, c_j). Moreover, we put $f_n(c_j) = f(c_j)$ for $j = 0, 1, \ldots, k$.
In this way we obtain a step function f_n such that

$$\|f_n - f\|_\infty \leqslant \frac{1}{n}$$

for $n = 1, 2, \ldots$. Hence we obtain that $\lim\limits_{n\to\infty} \|f_n - f\|_\infty = 0$.

Now, we are going to prove the implication $\Leftarrow$.
To this end assume that $f \in B([a, b])$ is a function being the limit of uniformly convergent sequence (f_n) of step functions on the interval $[a, b]$. Fix an arbitrary number $\varepsilon > 0$. Then we can find a natural number n such that

$$\|f_n - f\|_\infty \leqslant \frac{\varepsilon}{3}. \tag{4.1}$$

Keeping in mind the fact that f_n is a step function we deduce that for each $x \in (a, b)$ there exists an interval (c, d) containing x $\big($and, for $x = a$, there exists an interval (a, c), whereas for $x = b$ there exists an interval $(d, b)\big)$ such that

$$|f_n(t) - f_n(s)| \leqslant \frac{\varepsilon}{3}, \tag{4.2}$$

if $t, s \in (c, x)$ or if $t, s \in (x, d)$ (the situation is even simpler in the case of end intervals).

Further, let us take an arbitrary number $x \in [a, b]$ and next, let us choose an interval (c, x) or (x, d) such that for $t, s \in (c, x)$ $\big($or for $t, s \in (x, d)\big)$ the inequality (4.2) is satisfied.
In order to fix our attention assume that (c, x) is the desired interval. Then, for $t, s \in (c, x)$, in view of (4.1) and (4.2) we get

$$|f(t) - f(s)| \leqslant |f(t) - f_n(t)| + |f_n(t) - f_n(s)| + |f_n(s) - f(s)| \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

From the above inequality and Theorem 3.4 we conclude that there exists the left hand limit of the function f at the point x.
In the similar way we show that there exists the right hand limit $f(x+)$.
This means that $f \in R([a,b])$. □

Finally, taking into account the above proved theorem and Theorem 2.7 we infer that $R([a,b])$ is a closed subset of the space $B([a,b])$. Since the space $B([a,b])$ is complete, on the base of Theorem 2.1 we deduce that the space of regulated functions $R([a,b])$ is complete.
We formulate the above statement as the separated theorem.

Theorem 4.4. *The space of regulated functions $R([a,b])$ with the supremum norm is a Banach space.*

Observe that the space $R([a,b])$ is not separable. Indeed, fix arbitrarily $\varepsilon > 0$. For a fixed number $y \in [a,b]$ define the function f_y in the following way

$$f_y(x) = \begin{cases} \varepsilon & \text{for } x = y \\ 0 & \text{for } x \in [a,b],\ x \neq y. \end{cases}$$

Obviously $f_y \in R([a,b])$. Further notice that for $y_1, y_2 \in [a,b]$, $y_1 \neq y_2$, we have that $\|f_{y_1} - f_{y_2}\|_\infty = \varepsilon$. Hence, by virtue of Corollary 2.5 we obtain our claim. In what follows we pay our attention to points of discontinuity of regulated functions. At the beginning we recall the classification of points of discontinuity accepted in mathematical analysis [1, 7, 11].
Assume that $f : [a,b] \to \mathbb{R}$ is a given function.

Definition 4.5. Let $x_0 \in [a,b]$ be a point of discontinuity of the function f. We say that x_0 is *a point of discontinuity of the first kind* if there exist finite one-sided limits $f(x_0-) = \lim_{x \to x_0-} f(x)$, $f(x_0+) = \lim_{x \to x_0+} f(x)$. In the case $x_0 = a$ or $x_0 = b$ we assume the existence of finite limits $f(a+), f(b-)$, but it has to be satisfied the condition $f(a+) \neq f(a)$, $f(b-) \neq f(b)$, respectively.

Observe that if x_0 is a point of discontinuity of the first kind then the following situations may occur:

1° $f(x_0-) \neq f(x_0+)$.
In this case the point of discontinuity of the function f is called *a jump*.

2° $f(x_0-) = f(x_0+) \neq f(x_0)$.
In such a case we will say that f has at the point x_0 *a removable discontinuity.*

Definition 4.6. A point of discontinuity x_0 is called *a point of discontinuity of the second kind* if it is not a point of discontinuity of the first kind i.e., at least one of the one-sided limits is unbounded or does not exist.

Now, let us observe that if $f \in B([a,b])$ then f may have points of discontinuity of the first kind only. This implies that a regulated function on the interval $[a,b]$ has only points of discontinuity of the first kind. Thus, assume that $f \in R([a,b])$. Let us accept the following notation [1]:

$$D(f) = \big\{x \in [a,b] : f \text{ is discontinuous at } x\big\},$$
$$D_0(f) = \big\{x \in [a,b] : f \text{ has a removable discontinuity at } x\big\},$$
$$D_1(f) = \big\{x \in [a,b] : f \text{ has a jump at } x\big\}.$$

Observe that if $a \in D(f)$ $\big(b \in D(f)\big)$ then $a \in D_0(f)$ $\big(b \in D_0(f)\big)$.

It is worthwhile mentioning that if f is a monotone function on the interval $[a,b]$ then f has only points of discontinuity being jumps i.e., $D(f) = D_1(f)$. On the other hand it is well known that the set of points of discontinuity of each function which is monotone on the interval $[a,b]$ is at most countable.
The similar result holds also for any function $f \in R([a,b])$. Indeed, we have the following theorem [1].

Theorem 4.7. *The set of all points of discontinuity of any regulated function on the interval $[a,b]$ is at most countable.*

The proof of this result is given in [1] and we will not repeat it in this paper. However, we provide the proof of an analogous result in the next section for regulated functions on the interval $[a,b]$ with values in an arbitrary separable Banach space.

At the end of this section we provide a few facts indicating some special classes of regulated functions.
First of all let us pay our attention to the class of functions of bounded variation on the interval $[a,b]$ (cf. [1]). Let us denote this class by $BV([a,b])$.
Notice that $S([a,b]) \subset BV([a,b]) \subset B([a,b])$. Similarly as in the case of step functions it is easily seen that $BV([a,b])$ is a linear space over the field $\mathbb{R}$ (cf. [1]). Moreover, the space $BV([a,b])$ is not complete with respect to the supremum norm $\|\cdot\|_\infty$.

Further, let us pay attention to the fact that by the classical Jordan decomposition theorem we have that $BV([a,b]) \subset R([a,b])$.

The next important generalization of the space $BV([a,b])$ is the class of functions having the so-called bounded Wiener variation of order p (cf. [1]).
To define this class, fix a number $p \in [1,\infty)$. For a partition $P = \{x_0, x_1, \ldots, x_m\}$ of the interval $[a,b]$, where $a = x_0 < x_1 < \ldots < x_m = b$, we define the Wiener variation (of order p) of a function $f \in B([a,b])$ with respect to the partition P by putting

$$Var_p^W\big(f, P; [a,b]\big) = \sum_{i=1}^{m} \big|f(x_i) - f(x_{i-1})\big|^p.$$

The Wiener variation of order p of f is defined as

$$Var_p^W(f) = Var_p^W\big(f; [a,b]\big) = \sup\Big\{Var_p^W\big(f, P; [a,b]\big) : P \in \mathcal{P}([a,b])\Big\},$$

where the symbol $\mathcal{P}([a,b])$ stands for the set of all partitions of the interval $[a,b]$.
A function f $\big(f \in B([a,b])\big)$ is called *of bounded p-th variation* if $Var_p^W(f) < \infty$.

By the symbol $WBV_p([a,b])$ we denote the class of all functions of bounded Wiener p-th variation.

It can be shown [1] that $WBV_p([a,b])$ is a linear space and $BV([a,b]) \subset WBV_p([a,b]) \subset R([a,b])$ for any $p \geqslant 1$, but there are regulated functions which do not belong to the space $WBV_p([a,b])$.

In order to introduce the next class of functions denote by ϕ the so-called *Young function* i.e., $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\phi(0) = 0$, $\phi(t) > 0$ for $t > 0$, ϕ is continuous and convex on $\mathbb{R}_+$. Then, we can define the Wiener-Young variation of a function $f \in B([a,b])$ with respect to a partition $\mathcal{P}$ of the interval $[a,b]$ by putting

$$Var_\phi^W(f,P) = \sum_{i=1}^{m} \phi\big(|f(x_i) - f(x_{i-1})|\big).$$

The formula

$$Var_\phi^W(f) = \sup\big\{Var_\phi^W(f,P) : P \in \mathcal{P}([a,b])\big\}$$

defines *the Wiener-Young variation* of f on the interval $[a,b]$.

If we denote by $WBV_\phi([a,b])$ the class of functions of bounded Wiener-Young variation on $[a,b]$ then it can be shown that $WBV_\phi([a,b]) \subset R([a,b])$ and

$$\bigcup_{\phi \in \Phi} WBV_\phi([a,b]) = R([a,b]),$$

where Φ denotes the class of all Young functions [8, 9].

5. Generalizations

This section is devoted to show that the concept of a regulated function can be introduced in a more general setting.

Namely, let $(E, \|\cdot\|)$ be a real Banach space. We will denote by $B(x,r)$ the open ball in E centered at x and with radius r. The symbol $B([a,b],E)$ will denote the class of all functions acting from $[a,b]$ into E which are bounded on $[a,b]$. Obviously $B([a,b],E)$ forms a linear space and it can be equipped with the classical supremum norm $\|\cdot\|_\infty$ defined for $f \in B([a,b],E)$ in the following way

$$\|f\|_\infty = \sup\big\{\|f(x)\| : x \in [a,b]\big\}.$$

The space $B([a,b],E)$ is a Banach space.

Similarly, we can define the Banach space $C([a,b],E)$ consisting of all functions defined and continuous on the interval $[a,b]$ with values in E and with supremum norm $\|\cdot\|_\infty$.

Finally, let us consider the set $R([a,b],E)$ consisting of all regulated functions $f : [a,b] \to E$. In other words, $f \in R([a,b],E)$ if $f \in B([a,b],E)$ and the function f has one-sided limits at every point $x \in (a,b)$ and if it has the right hand limit $f(a+)$ and

the left hand limit $f(b-)$.
In the same way as in Section 4 we can show that the set $R([a, b], E)$ forms a linear space with usual operations on functions. Moreover, if we introduce in $R([a, b], E)$ the supremum norm $\| \cdot \|_\infty$ then we can prove that $R([a, b], E)$ is a Banach space. The proof of this theorem can be conducted exactly in the same way as the proof of Theorem 4.4 and, similarly as before, the space of step functions $S([a, b], E)$ plays the key role in argumentation of that proof. Therefore, we will not repeat details of the proof.

In this section we restrict ourselves to prove an analogon of Theorem 4.7 in the case of the space of regulated functions $R([a, b], E)$. To adopt some reasonings utilized in the proof of Theorem 4.7 given in [1], we will assume that the Banach space E is separable. Nevertheless, we provide here the details of the complete proof of the announced theorem to fill in some gaps occuring in the suitable proof given in [1].

Theorem 5.1. *Let E be a separable Banach space. Then the set of all points of discontinuity of an arbitrary regulated function $f \in R([a, b], E)$ is at most countable.*

Proof. Observe that we can restrict ourselves to the set of points of discontinuity of f belonging to the interval (a, b), since the set of points of discontinuity of f on the interval $[a, b]$ can differ only of at most two points. Further, consider the average function $\overline{f}$ of f defined in the following way

$$\overline{f}(x) = \begin{cases} \frac{1}{2}\big(f(x-) + f(x+)\big) & \text{for } x \in D_1(f) \\ f(x) & \text{otherwise.} \end{cases}$$

Obviously we have that $D(\overline{f}) = D(f)$, $D_0(\overline{f}) = D_0(f)$, $D_1(\overline{f}) = D_1(f)$, where the sets $D(f)$, $D_0(f)$ and $D_1(f)$ were defined earlier. Thus, it is sufficient to show that the set $D(\overline{f})$ is at most countable.

To this end assume first that $x_0 \in D_0(f)$. Then $f(x_0-) = f(x_0+) \neq f(x_0)$. Take the number

$$\varepsilon = \frac{1}{2}\big\|f(x_0) - f(x_0+)\big\|.$$

Taking into account the definition of the one-sided limit of a function we can find a number $\delta > 0$ such that for $x \in (x_0 - \delta, x_0)$ we have

$$\|f(x) - f(x_0-)\| < \varepsilon \tag{5.1}$$

and for $x \in (x_0, x_0 + \delta)$ we have

$$\|f(x) - f(x_0+)\| < \varepsilon. \tag{5.2}$$

Now, let us observe that for $x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$ the following inequality holds

$$\|f(x) - f(x_0)\| > \varepsilon. \tag{5.3}$$

To prove this inequality suppose contrary. Then, for some $x \in (x_0-\delta, x_0) \cup (x_0, x_0+\delta)$ we have

$$\|f(x) - f(x_0)\| \leqslant \varepsilon. \tag{5.4}$$

Now, in view of (5.1), (5.2) and (5.4), we get

$$\|f(x_0) - f(x_0+)\| \leqslant \|f(x_0) - f(x))\| + \|f(x) - f(x_0+)\| < 2\varepsilon,$$

which contradicts to the choice of the number ε.
This shows that in the ball $B\big((x_0, f(x_0)), r\big)$ in the Banach space $[a,b] \times E$ (with the maximum norm, for example), where $r = \min\{\delta, \varepsilon\}$, there are no points of the graph of the function f except the point $(x_0, f(x_0))$. Thus the point $(x_0, f(x_0))$ is an isolated point of the graph of the function f (or $\overline{f}$).

Now, assume that $x_0 \in D_1(f)$. Then $f(x_0-) \neq f(x_0+)$. Let us put $\varepsilon = \frac{1}{4}\|f(x_0+) - f(x_0-)\|$. Taking into account the definition of one-sided limits let us find the number $\delta > 0$ such that for $x \in (x_0 - \delta, x_0)$ the following inequality is satisfied

$$\|f(x) - f(x_0-)\| < \varepsilon, \tag{5.5}$$

while for $x \in (x_0, x_0 + \delta)$ we have

$$\|f(x) - f(x_0+)\| < \varepsilon. \tag{5.6}$$

Further, observe that for $x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$ the following inequality holds

$$\|f(x) - \overline{f}(x_0)\| > \varepsilon. \tag{5.7}$$

Indeed, suppose contrary. Then, for some $x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$ we have

$$\|f(x) - \overline{f}(x_0)\| \leqslant \varepsilon. \tag{5.8}$$

Then, linking inequalities (5.5), (5.6) and (5.8) we get

$$\|\overline{f}(x_0) - f(x_0+)\| \leqslant \|\overline{f}(x_0) - f(x)\| + \|f(x) - f(x_0+)\| < \varepsilon + \varepsilon = 2\varepsilon. \tag{5.9}$$

On the other hand, we have

$$\big\|\overline{f}(x_0) - f(x_0+)\big\| = \left\|\frac{1}{2}\big(f(x_0-) + f(x_0+)\big) - f(x_0+)\right\| = \frac{1}{2}\big\|f(x_0-) - f(x_0+)\big\| = 2\varepsilon.$$

Thus we obtain the contradiction with inequality (5.9). This proves inequality (5.7).

Further, consider the ball $B\big((x_0, \overline{f}(x_0)), r\big)$ in the space $[a,b] \times E$ (considered earlier), where $r = \min\{\delta, \varepsilon\}$. Then from inequality (5.7) we conclude that this ball contains no points of the graph of the function $\overline{f}$ except the point $(x_0, \overline{f}(x_0))$. Hence we deduce that $(x_0, \overline{f}(x_0))$ is an isolated point of the function $\overline{f}$.

Summing up, we proved that the set H of all points of the graph of the function $\overline{f}$, which are the centers of the described balls, consists of only isolated points. Thus, according to the earlier given definition, the set H is an isolated set in the space $[a,b] \times E$ with the maximum norm. Obviously the mentioned space is separable. Hence, in view of Theorem 2.4 we infer that the set H is at most countable. This leads to the final conclusion that the set $D(\overline{f})$ $\big($and the set $D(f)\big)$ is at most countable. □

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Existence and Controllability Results for Sobolev-type Fractional Impulsive Stochastic Differential Equations with Infinite Delay

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Abstract: In this paper, we prove the existence of mild solutions for Sobolev-type fractional impulsive stochastic differential equations with infinite delay in Hilbert spaces. In addition, the controllability of the system with nonlocal conditions and infinite delay is studied. An example is provided to illustrate the obtained theory.

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Keywords and Phrases: Fractional impulsive stochastic differential equations; Fixed point principle; Mild solution; Controllability.

1. Introduction

Stochastic differential equations is an important emerging field and has attracted great interest from both theoretical and applied disciplines, which has been successfully applied to problems in physics, biology, chemistry, mechanics and so on (see [6, 7, 9, 14, 21]) and the references therein). In the present literature, there are many papers on the existence and uniqueness of solutions to stochastic differential equations (see [2, 3, 25]). The stochastic differential equations with infinite delay have become important in recent years as mathematical models of phenomena in both physical and social sciences [26, 28]. The existence of mild solutions and (approximate) controllability for different types of fractional evolution systems have been reported by many researchers (see [5, 17, 19, 25, 27, 28, 29] and the references therein).

The Sobolev type (fractional) equation appears in a variety of physical problems such as flow of fluid through fissured rocks, thermodynamics, propagation of long waves of small amplitude and shear in second order fluids and so on [20]. There are many interesting results on the the existence and uniqueness of mild solutions and approximate controllability for a class of Sobolev type fractional evolution equations, we refer the reader to [1, 11, 15, 18, 20].

Recently, the existence of mild solutions and the approximate controllability of fractional Sobolev type evolution system in Banach spaces have been studied in many publications (see [11, 12, 13, 18, 15] and the references therein).

More recently, Benchaabane and Sakthivel [4] investigated the existence and uniqueness of mild solutions for a class of nonlinear fractional Sobolev type stochastic differential equations in Hilbert spaces. A new set of sufficient condition is established with the coefficients in the equations satisfying some non-Lipschitz conditions. Revathi et al. [22] studied the local existence of mild solution for a class of stochastic functional differential equations of Sobolev-type with infinite delay. The results are extended to study the local existence results for neutral stochastic differential equations of Sobolev-type.

For our knowledge, there is no work reported on Sobolev-type fractional impulsive stochastic differential equation with infinite delay. Motivated by the above works, the purpose of this paper is to prove the existence and uniqueness of mild solutions and the controllability for the Sobolev-type fractional impulsive stochastic differential equation with infinite delay. Our approach is based on the fixed point theorem. The rest of this paper is organized as follows. In Section 2, we will provide some basic definitions, lemmas and basic properties of fractional calculus. The concept of mild solutions, a set of sufficient conditions for the existence and uniqueness of mild solutions for the considered equations is obtained in Section 3. In Section 4, provide a sufficient condition for the controllability for a class of fractional evolution equations of Sobolev-type impulsive stochastic fractional equations with nonlocal conditions and infinite delay.

2. Preliminaries and basic properties

Let $\mathcal{H}, \mathcal{K}$ be two separable Hilbert spaces and $L(\mathcal{K}, \mathcal{H})$ be the space of bounded linear operators from $\mathcal{K}$ into $\mathcal{H}$. For convenience, we will use the same notation $\| \,.\, \|$ to denote the norms in $\mathcal{H}, \mathcal{K}$ and $L(\mathcal{K}, \mathcal{H})$, and use $(.,.)$ to denote the inner product of $\mathcal{H}$ and $\mathcal{K}$ without any confusion. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. $\omega = (\omega_t)_{t\geq 0}$ be a Q-Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the covariance operator Q such that $TrQ < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathcal{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \cdots$, and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(\omega(t), e)_{\mathcal{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k} (e_k, e)_{\mathcal{K}} \beta_k(t), \quad e \in \mathcal{K}, \quad t \geq 0 \,.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(\mathbb{Q}^{\frac{1}{2}}\mathcal{K}, \mathcal{H})$ be the space of all HilbertSchmidt operators from $\mathbb{Q}^{\frac{1}{2}}\mathcal{K}$ to $\mathcal{H}$ with the inner product $< \varphi, \psi >_{\mathscr{L}_2^0} = Tr[\varphi Q \psi^*]$.

In this paper, we consider the following Sobolev-type fractional impulsive stochastic differential equations with infinite delay:

$$\begin{cases} D_t^\alpha Lx(t) = Ax(t) + f(t, x_t)) + \sigma(t, x_t)\dfrac{d\omega(t)}{dt}, & t \in J = [0,T], T > 0, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad k = 1, \cdots, m, \\ x(t) = \phi, \quad \phi \in B_h, \end{cases} \tag{2.1}$$

where D_t^α is the Caputo fractional derivative of order α, $\frac{1}{2} < \alpha < 1$, $x(.)$ takes the value in a separable Hilbert space $\mathcal{H}$. We assume that the operators L and A are defined as: $A : D(A) \in \mathcal{H} \to \mathcal{H}$ and $L : D(L) \in \mathcal{H} \to \mathcal{H}$ generates a strongly continuous semigroup $S(t)_{t \geq 0}$. Here $0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq t_{m+1} = T$, $\Delta x(t_k) = I_k(x(t_k^-)) = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0} x(t_k - h)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively. The initial data $\phi = \{\phi(t); t \in (-\infty, 0]\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}_h$-valued random variable independent of ω with finite second moments. Further $f : J \times \mathcal{B}_h \to \mathcal{H}$ and $\sigma : J \times \mathcal{B}_h \to \mathcal{L}_2^0(\mathcal{K}, \mathcal{H})$ are appropriate mappings will be specified later.

We introduce the following assumptions on the operators L and A.

L1 L and A are closed linear operators,

L2 $D(L) \subset D(A)$ and L is bijective,

L3 $L^{-1} : \mathcal{H} \to D(L)$ is compact.

Remark 2.1. From $(L3)$, we deduce that L^{-1} is a bounded operators, for short, we denote by $C_1 = \|L^{-1}\|$ and $C_2 = \|L\|$. Note $(L3)$ also implies that L is closed since the fact: L^{-1} is closed and injective, then its inverse is also closed. It comes from $(L1) - (L3)$ and the closed graph theorem, we obtain the boundedness of the linear operator $AL^{-1} : \mathcal{H} \to \mathcal{H}$. Consequently, $-AL^{-1}$ generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ in $\mathcal{H}$. We suppose that $M := \max_{t\in[0,T]} \|S(t)\|$.

Now, we present the abstract space phase $\mathcal{B}_h$. Assume that $h : (-\infty, 0] \to (0, +\infty)$ with $l = \int_{-\infty}^0 h(t)dt < \infty$ a continuous function. We define the abstract phase space $\mathcal{B}_h$ by

$$\mathcal{B}_h := \Bigg\{ \phi : (-\infty, 0] \times \Omega \to \mathcal{H}, \text{for any} \quad a > 0, (E \mid \phi(\theta \mid^2)^{\frac{1}{2}}$$

$$\text{is bounded and measurable function on} \quad [-a, 0] \quad \text{and}$$

$$\int_{-\infty}^0 h(s) \sup_{s\leq\theta\leq 0} (E\|\phi(\theta\|^2)^{\frac{1}{2}} < +\infty \Bigg\}.$$

If $\mathcal{B}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{B}_h} := \int_{-\infty}^0 h(s) \sup_{s\leq\theta\leq 0} (E\|\phi(\theta)\|^2)^{\frac{1}{2}}, \phi \in \mathcal{B}_h,$$

then $(\mathcal{B}_h, \|.\|_{\mathcal{B}_h})$ is a Banach space [23, 24].

Now we consider the space

$$\mathcal{B}_T := \Bigg\{ x : (-\infty, T] \times \Omega \to \mathcal{H}, \quad \text{such that} \quad x|_{J_k} \in C(J_k, \mathcal{H})$$

and there exist $\quad x(t_k^+), \quad$ and $\quad x(t_k^-) \quad$ with $\quad x(t_k) = x(t_k^-), x_0 = \phi \in \mathcal{B}_h, k = 1, \ldots, m$

$$\text{and} \quad \sup_{0 \le s \le T} (E\|x(s)\|^2) < \infty \Bigg\},$$

where $x|_{J_k}$ is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$. We endow a seminorm $\|.\|_{\mathcal{B}_T}$on $\mathcal{B}_T$, it is defined by

$$\|x\|_{\mathcal{B}_T} = \|\phi\|_{\mathcal{B}_h} + \sup_{0 \le s \le T} (E\|x(s)\|^2)^{\frac{1}{2}}, x \in \mathcal{B}_T.$$

We recall the following lemma:

Lemma 2.2. [*24*] *Assume that $x \in \mathcal{B}_T$; then for $t \in J, x_t \in \mathcal{B}_h$. Moreover*

$$l(E\|x(t)\|^2)^{\frac{1}{2}} \le l \sup_{s \in [0,t]} E\|x(s)\|^2)^{\frac{1}{2}} + \|x_0\|_{\mathcal{B}_h} ,$$

where $l = \int_{-\infty}^{0} h(s)ds < \infty$.

Definition 2.3. [8] The Caputo derivative of order α for a function $f : [0, \infty) \to \mathbb{R}$, which is at least n-times differentiable can be defined as

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = I_a^{(n-\alpha)} \left(\frac{d^n f}{dt^n} \right)(t) \tag{2.2}$$

for $n - 1 \le \alpha < n, n \in \mathbb{N}$. If $0 < \alpha \le 1$, then

$$D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} \left(\frac{df(s)}{ds} \right) ds \ . \tag{2.3}$$

Obviously, the Caputo derivative of a constant is equal to zero.

Definition 2.4. The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (s-t)^{\alpha-1} f(s) ds \tag{2.4}$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where the Γ is the gamma function.

Remark 2.5. If $f$is an abstract function with values in $\mathcal{H}$, then integrals which appear in Definition 2.4 are taken in Bochners sense.

For $x \in \mathcal{H}$, we define two families $\{T_L(t), t \geq 0\}$ and $\{S_L(t), t \geq 0\}$ of operators by

$$\begin{aligned} T_L(t) &:= T_\alpha(t)L^{-1} = \textstyle\int_0^\infty L^{-1}\Psi_\alpha(\theta)S(t^\alpha\theta)d\theta, \\ S_L(t) &:= S_\alpha(t)L^{-1} = \alpha \textstyle\int_0^\infty L^{-1}\theta\Psi_\alpha(\theta)S(t^\alpha\theta)d\theta, \end{aligned} \tag{2.5}$$

where

$$\Psi_\alpha(\theta) := \frac{1}{\pi\alpha}\sum_{n=1}^{\infty}(-\theta)^{n-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha), \quad \theta \in]0, +\infty[\tag{2.6}$$

is a probability density function defined on $]0, \infty[$, which satisfies that $\Psi_\alpha(\theta) \geq 0$ and $\int_0^\infty \Psi_\alpha(\theta)d\theta = 1$.

Lemma 2.6. *[30] The operators T_α and S_α have the following properties:*

1. *For any fixed $x \in \mathcal{H}$, $\|T_\alpha(t)x\| \leq M\|x\|$, $\|S_\alpha(t)x\| \leq \dfrac{M}{\Gamma(\alpha)}\|x\|$.*

2. *$\{T_\alpha(t), t \geq 0\}$ and $\{S_\alpha(t), t \geq 0\}$ are strongly continuous.*

Lemma 2.7. *[13] The operators T_L and S_L defined by (2.5) have the following properties:*

1. *For any fixed $t \geq 0$, $T_L(t)$ and $S_L(t)$ are linear and bounded operators, and for any $x \in \mathcal{H}$*

$$\begin{aligned} \|T_L(t)x\| &\leq MC_1\|x\|, \\ \|S_L(t)x\| &\leq \frac{MC_1}{\Gamma(\alpha)}\|x\|. \end{aligned} \tag{2.7}$$

2. *$\{T_L(t), t \geq 0\}$ and $\{S_L(t), t \geq 0\}$ are compact.*

The key tool in our approach is the following form of the Krasnoselskii's fixed point theorem [10].

Theorem 2.8. *Let B be a nonempty closed convex of a Banach space $(X, \|.\|)$. Suppose that P and Q map B into X such that*

(i) $Px + Qy \in B$ whenever $x, y \in B$;

(ii) P is compact and continuous;

(iii) Q is a contraction mapping.

Then there exists $z \in B$ such that $z = Pz + Qz$.

3. Existence of mild solutions

In this section, we first establish the existence of mild solutions to Sobolev-type fractional stochastic differential equations (2.1). More precisely, we will formulate and prove sufficient conditions for the existence of solutions to (2.1) with infinite delay and impulses. First, we first define the concept of mild solution to our problem.

Definition 3.1. A cádlàg $\mathcal{H}$-valued process x is said to be a mild solution of (2.1) if

1. $x(t)$ is $\mathcal{F}_t$-adapted and $\{x(t), t \in [0,T]\}$ is $\mathcal{B}_h$-valued,
2. for each $t \in J$, $x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) = \;& T_L(t)L\phi(0) + \int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, x_s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_L(t-s)\sigma(s, x_s) d\omega(t) \\ &+ \sum_{0<t_k<t} T_L(t-t_k) I_k(x(t_k^-)), \end{aligned} \tag{3.1}$$

3. $x(t) = \phi(t)$ on $(-\infty, 0]$ satisfying $\|\phi\|^2_{\mathcal{B}_h} < \infty$.

Let us introduce the following hypotheses:

(H1) There exists $L_f > 0$ such that

$$E\|f(t,x) - f(t,y)\|^2_{\mathcal{H}} \leq L_f \|x-y\|^2_{\mathcal{B}_h}, \quad t \geq 0.$$

(H2) There exists $L_\sigma > 0$ such that

$$E\|\sigma(t,x) - \sigma(t,y)\|^2_{\mathscr{L}_2^0} \leq L_\sigma \|x-y\|^2_{\mathcal{B}_h}, \quad t \geq 0.$$

(H3) For all $x \in \mathcal{H}$, there exist constants $L_k > 0$, $k = 1, \ldots, m, \ldots$ for each

$$|I_k(y)|^2 \leq L_k \; .$$

Theorem 3.2. *Assume that* $f(t,0) = \sigma(t,0) = 0$, $\forall t \geq 0$. *Assume that hypotheses* $(H1) - (H3)$ *hold. If*

$$r \geq 3M^2C_1^2 \sum_{k=1}^{m} L_k + \frac{3M^2C_1^2T^{2\alpha}\chi}{\Gamma^2(\alpha)} \left[\frac{L_f}{\alpha^2} + \frac{L_\sigma}{T(2\alpha-1)}\right] \tag{3.2}$$

and

$$\frac{2M^2C_1^2}{\Gamma^2(\alpha)} T^{2\alpha} \left[\frac{L_f l}{\alpha^2} + \frac{L_\sigma l}{T(2\alpha-1)}\right] < 1, \tag{3.3}$$

then system (2.1) *has a mild solution on* $(-\infty, T]$.

Proof. Transform the problem (2.1) into a fixed-point problem. Consider the operator $\Psi : \mathcal{B}_T \to \mathcal{B}_T$ defined by

$$\Psi x(t) = \begin{cases} \phi(t), \quad \text{if} \quad t \in (-\infty, 0], \\ T_L(t)L\phi(0) + \displaystyle\int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, x_s) ds \\ + \displaystyle\int_0^t (t-s)^{\alpha-1} S_L(t-s)\sigma(s, x_s) dW(t) \\ + \displaystyle\sum_{0<t_k<t} T_L(t-t_k) I_k(x(t_k^-)). \end{cases}$$

For $\phi \in \mathcal{B}_h$, we define $\widehat{\phi}$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ T_L(t)L\phi(0), & t \in [0, +\infty[; \end{cases} \quad \text{then } \widehat{\phi} \in \mathcal{B}_T.$$

Let $x(t) = y(t) + \widehat{\phi}(t), -\infty < t < T$.
It is evident that y satisfies that $y_0 = 0, t \in (-\infty, 0]$ and

$$\begin{aligned} y(t) &= \int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, y_s + \widehat{\phi}_s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_L(t-s)\sigma(s, y_s + \widehat{\phi}_s) d\omega(t) \\ &+ \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)), \ t \in J \end{aligned}$$

if and only if x satisfies that $x(t) = \phi(t), t \in (-\infty, 0]$, and

$$\begin{aligned} x(t) = T_L(t)L\phi(0) \quad &+ \int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, x_s) ds \\ &+ \int_0^t (t-s)^{\alpha-1} S_L(t-s)\sigma(s, x_s) d\omega(t) \\ &+ \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(x(t_k^-)), \ t \in J \ . \end{aligned}$$

Set $\mathcal{B}_T^0 = \{z \in \mathcal{B}_T, \quad \text{such that} \quad z_0 = 0\}$ and for any $z \in \mathcal{B}_T^0$ we have

$$\|z\|_{\mathcal{B}_T^0} = \|z_0\|_{\mathcal{B}_h} + \sup_{t \in J}(E\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t \in J}(E\|z(t)\|^2)^{\frac{1}{2}},$$

where $\|z_0\|_{\mathcal{B}_h} = 0$. Thus $(\mathcal{B}_T^0, \|.\|_{\mathcal{B}_T^0})$ is a Banach space.

Let the operator $\widehat{\Psi} : \mathcal{B}_T^0 \longrightarrow \mathcal{B}_T^0$ be defined by

$$\widehat{\Psi} y(t) = \begin{cases} 0, \quad t \in (-\infty, 0] \\ \displaystyle\int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, y_s + \widehat{\phi}_s) ds \\ + \displaystyle\int_0^t (t-s)^{\alpha-1} S_L(t-s)\sigma(s, y_s + \widehat{\phi}_s) d\omega(t) \\ + \displaystyle\sum_{0<t_k<t} T_L(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)), \ t \in J. \end{cases}$$

Set $\mathcal{B}_r = \{y \in \mathcal{B}_T^0, \quad \|y\|^2_{\mathcal{B}_T} \le r,\ r > 0\}$. The set $\mathcal{B}_r$ is clearly a bounded closed convex set in $\mathcal{B}_T^0$ for each $r > 0$ and $y \in \mathcal{B}_r$, by Lemma 2.2 we have

$$\begin{aligned}\|y_t + \widehat{\phi}_t\|^2_{\mathcal{B}_h} \le\ & 2(\|y_t\|^2_{\mathcal{B}_h} + \|\widehat{\phi}_t\|^2_{\mathcal{B}_h}) \\ & \le 4(l^2 \sup_{s\in[0,t]} E\|y(s)\|^2_{\mathcal{H}} + \|y_0\|^2_{\mathcal{B}_h}) \\ & +4(l^2 \sup_{s\in[0,t]} E\|\widehat{\phi}(s)\|^2_{\mathcal{H}} + \|\widehat{\phi}_0\|^2_{\mathcal{B}_h}) \\ & \le 4\|\phi\|^2_{\mathcal{B}_h} + 4l^2(r + M^2 C_1^2 C_2^2 E\|\phi(0)\|^2_{\mathcal{H}}) = \chi\ .\end{aligned}$$

Now, let the two operators $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$ be defined as

$$\widehat{\Psi}_1 y(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \displaystyle\sum_{0<t_k<t} T_L(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)), & t \in [0,T], \end{cases} \tag{3.4}$$

and

$$(\widehat{\Psi}_2 y)(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ \displaystyle\int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, y_s + \widehat{\phi}_s) ds & \\ \displaystyle + \int_0^t (t-s)^{\alpha-1} S_L(t-s) \sigma(s, y_s + \widehat{\phi}_s) d\omega(t), & t \in [0,T]. \end{cases} \tag{3.5}$$

It is clear that

$$\widehat{\Psi}_1 + \widehat{\Psi}_2 = \Psi\ .$$

Then, the problem of finding a solution of (2.1) is reduced to finding a solution of the operator equation $y(t) = \widehat{\Psi}_1(y)(t) + \widehat{\Psi}_2(y)(t), t \in (-\infty, T]$. In order to use Theorem 2.8, we will verify that $\widehat{\Psi}_1$ is compact and continuous while $\widehat{\Psi}_2$ is a contraction operator.

For the sake of convenience, we divide the proof into several steps.

Step 1. We show that $\widehat{\Psi}_1 y + \widehat{\Psi}_2 y^* \in \mathcal{B}_r$, for $y, y^* \in \mathcal{B}_r$. For $t \in [0,T]$, we have

$$\begin{aligned}
\|(\widehat{\Psi}_1 y)(t)+(\widehat{\Psi}_2 y^*)(t)\|^2_{\mathcal{H}} \leq 3 &\sum_{0<t_k<t} \left\| T_L(t-t_k)\|^2 E\|I_k(y(t_k^-)+\widehat{\phi}(t_k^-))\right\|^2_{\mathcal{H}} \\
&+3E\left\|\int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, y_s^* + \widehat{\phi}_s)ds\right\|^2_{\mathcal{H}} \\
&+3E\left\|\int_0^t (t-s)^{\alpha-1} S_L(t-s) \sigma(s, y_s^* + \widehat{\phi}_s)dW(t)\right\|^2_{\mathcal{H}} \\
\leq & 3M^2C_1^2 \sum_{k=1}^{m} L_k \\
&+3\int_0^t \|(t-s)^{\alpha-1} S_L(t-s)\|^2 E\|f(s, y_s^*+\widehat{\phi}_s)\|^2_{\mathcal{H}} ds \\
&+3\int_0^t \|(t-s)^{\alpha-1} S_\alpha(t-s)\|^2 E\|\sigma(s, y_s^*+\widehat{\phi}_s)\|^2_{\mathcal{H}} ds \\
\leq & \frac{3M^2C_1^2}{\Gamma^2(\alpha)} \frac{T^\alpha}{\alpha} \\
&\int_0^t (t-s)^{\alpha-1}\ ds + \frac{2M^2C_1^2 L_\sigma \chi}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2(\alpha-1)} ds \\
\leq & 3M^2C_1^2 \sum_{k=1}^{m} L_k + \frac{3M^2C_1^2 L_f \chi}{\Gamma^2(\alpha)} \frac{T^{2\alpha}}{\alpha^2} + \frac{2M^2C_1^2 L_\sigma \chi}{\Gamma^2(\alpha)} \frac{T^{2\alpha-1}}{2\alpha-1} \\
= & 3M^2C_1^2 \sum_{k=1}^{m} L_k + \frac{3M^2C_1^2 T^{2\alpha}\chi}{\Gamma^2(\alpha)} \left[\frac{L_f}{\alpha^2} + \frac{L_\sigma}{T(2\alpha-1)}\right].
\end{aligned}$$

Then

$$\|(\widehat{\Psi}_1 y)(t)+(\widehat{\Psi}_2 y^*)(t)\|^2_{\mathcal{H}} \leq 3M^2C_1^2 \sum_{k=1}^{m} L_k + \frac{3M^2C_1^2T^{2\alpha}\chi}{\Gamma^2(\alpha)}\left[\frac{L_f}{\alpha^2}+\frac{L_\sigma}{T(2\alpha-1)}\right] \leq r\ .$$

Hence, we get $\widehat{\Psi}_1 y + \widehat{\Psi}_2 y^* \in \mathcal{B}_r$.

Step 2. The map $\widehat{\Psi}_1$ is continuous on $\mathcal{B}_r$.

Let $\{y^n\}_{n=1}^\infty$ be a sequence in $\mathcal{B}_r$ with $\lim y^n \to y \in \mathcal{B}_r$. Then for $t \in J$ we have

$$E\|(\widehat{\Psi}_1 y^n)(t) - (\widehat{\Psi}_1 y)(t)\|^2_{\mathcal{H}} \leq$$
$$\sum_{0<t_k<t} \|T_\alpha(t-t_k)\|^2 E\|I_k(y^n(t_k^-)+\widehat{\phi}(t_k^-)) - I_k(y(t_k^-)+\widehat{\phi}(t_k^-))\|^2_{\mathcal{H}}\ .$$

Since the functions $I_i, i=1,2,\dots,m$ are continuous hence $\lim_{n\to\infty} \|(\widehat{\Psi}_1 y^n)(t) - (\widehat{\Psi}_1 y)(t)\|^2_{\mathcal{H}} = 0$ which implies that the mapping $\widehat{\Psi}_1$ is continuous on $\mathcal{B}_r$.

Step 3. $\widehat{\Psi}_1$ maps bounded sets into bounded sets in $\mathcal{B}_r$.

Let us prove that for $r > 0$ there exists a $\widehat{r} > 0$ such that for each $y \in \mathcal{B}_r$ we have $E\|(\widehat{\Psi}_1 y)(t)\|^2_{\mathcal{H}} < \widehat{r}$ for $t \in J$.

$$\begin{aligned} E\|(\widehat{\Psi}_1 y)(t)\|^2_{\mathcal{H}} &\leq \sum_{0<t_k<t} \|T_L(t-t_k)\|^2 E\|I_k(y(t_k^-) + \widehat{\phi}(t_k^-))\|^2_{\mathcal{H}} \\ &\leq M^2 C_1^2 \textstyle\sum_0^m L_k = \widehat{r}\ , \end{aligned}$$

which proves the desired result.

Step 4. The map $\widehat{\Psi}_1$ is equicontinuous.

Let $u,\, v \in J$, $0 \leq u < v \leq T$, $y \in \mathcal{B}_r$, we obtain

$$E\|(\widehat{\Psi}_1 y)(v) - (\widehat{\Psi}_1 y)(u)\|^2_{\mathcal{H}} \leq C_1^2 \sum_{0<t_k<u} L_k \|T_\alpha(v-t_k) - T_\alpha(u-t_k)\|^2.$$

The right-hand side tends to zero as $v - u \longrightarrow 0$, since T_α is strongly continuous and it allows us to conclude that

$$\lim_{u\to v} \|T_\alpha(v-t_k) - T_\alpha(u-t_k)\|^2 = 0,$$

which implies that $\widehat{\Psi}_1(\mathcal{B}_r)$ is equicontinuous.

Finally, combining Step 1 to Step 4 together with Ascoli's theorem, we conclude that the operator $\widehat{\Psi}_1$ is compact.

Step 5. $\widehat{\Psi}_2$ is a contraction mapping.

Let $y,\, y^* \in \mathcal{B}_r$ and $t \in J$ we have

$$\begin{aligned} &E\|(\widehat{\Psi}_2 y)(t) - (\widehat{\Psi}_2 y^*)(t)\|^2_{\mathcal{H}} \\ &\quad \leq 2E\left\| \int_0^t (t-s)^{\alpha-1} S_L(t-s) \left[f(s, y_s + \widehat{\phi}_s) - f(s, y_s^* + \widehat{\phi}_s) \right] ds \right\|^2_{\mathcal{H}} \\ &\qquad + 2E\left\| \int_0^t (t-s)^{\alpha-1} S_L(t-s) \left[\sigma(s, y_s + \widehat{\phi}_s) - \sigma(s, y_s^* + \widehat{\phi}_s) \right] d\omega(s) \right\|^2_{\mathcal{H}} \\ &\quad \leq 2\int_0^t \|(t-s)^{\alpha-1} S_L(t-s)\| ds \int_0^t \|(t-s)^{\alpha-1} S_L(t-s)\| \\ &\qquad \times E\|f(s, y_s + \widehat{\phi}_s) - f(s, y_s^* + \widehat{\phi}_s)\|^2_{\mathcal{H}} ds \\ &\qquad + 2\int_0^t \|(t-s)^{\alpha-1} S_L(t-s)\|^2 E\|\sigma(s, y_s + \widehat{\phi}_s) - \sigma(s, y_s^* + \widehat{\phi}_s)\|^2_{\mathcal{H}} ds \end{aligned}$$

$$
\begin{aligned}
\leq & 2\frac{M^2C_1^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{\alpha-1}ds\int_0^t (t-s)^{\alpha-1}L_f\|y(s)-y^*(s)\|^2_{\mathcal{B}_h}ds \\
& +\frac{2M^2C_1^2}{\Gamma^2(\alpha)}\int_0^t (t-s)^{2(\alpha-1)}L_\sigma\|y_s-y_s^*\|^2_{\mathcal{B}_h}ds \\
\leq & \frac{2M^2C_1^2}{\Gamma^2(\alpha)}T^{2\alpha}\left[\frac{L_f l}{\alpha^2}+\frac{L_\sigma l}{T(2\alpha-1)}\right]\|y-y^*\|^2_{\mathcal{B}_T^0} .
\end{aligned}
$$

By the condition (3.3), we obtain that $\widehat{\Psi}_2$ is a contraction mapping. Hence, by Krasnoselskii's fixed point theorem we can conclude that the problem (2.1) has at least one solution on $(-\infty, T]$. This completes the proof of the theorem.

□

Example 3.3. In this section, we consider an example to illustrate our main theorem. We examine the existence of solutions for the following fractional stochastic partial differential equation of the form

$$
\begin{cases}
D_t^\alpha[z(t,x)-z_{xx}(t,x)] = z_{xx}(t,x)+F(t,z(t-R,x)) \\
\qquad\qquad + G(t,z(t-R,x))\dfrac{d\omega(t)}{dt}, x\in=[0,\pi], R>0, t\neq t_k, \\
I_k(z(t_k^-,x)) = z(t_k^+,x)-z(t_k^-,x), \quad k=1,\dots,n, \\
z(t,x)=\phi(t,x), \quad t\in(-\infty,0], x\in[0,\pi], \\
z(t,0)=0=z(t,\pi), t\geq 0,
\end{cases}
\tag{3.6}
$$

where $\omega(t)$ is a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$; D_t^α is the Caputo fractional derivative of order $0<\alpha<1$; $0<t_1<t_2<\cdots<t_n<T$ are prefixed numbers.

Let $\mathcal{K}=\mathcal{H}=L^2([0,\pi])$ with the norm $\|\cdot\|$. Define the operators $A: D(A)\subset \mathcal{H}\longrightarrow\mathcal{H}$, $L: D(L)\subset\mathcal{H}\longrightarrow\mathcal{H}$, by $Az=-z^{''}$ and $Lz=z-z^{''}$, where each domain $D(A)$ and $D(L)$ is given by $\{z\in\mathcal{H}, z, z^{'}$ are absolutely continuous, $z^{''}\in \mathcal{H}$ and $z(0)=z(\pi)=0\}$.

Further, A and L can be $Az=\sum_{n=1}^{\infty} n^2(z,z_n)z_n$, $z\in D(A)$, $Lz=\sum_{n=1}^{\infty}(1+n^2)(z,z_n)z_n$, $z\in D(L)$, where $z_n(x)=\sqrt{\frac{2}{\pi}}\sin(nx), n=1,2,\dots,$ is the orthogonal set of vectors of A. Also, for $z\in\mathcal{H}$ $L^{-1}z=\sum_{n=1}^{\infty}\frac{1}{(1+n^2)}(z,z_n)z_n$, $AL^{-1}z=\sum_{n=1}^{\infty}\frac{n}{(1+n^2)}(z,z_n)z_n$, $T(t)z=\sum_{n=1}^{\infty} exp\frac{n^2t}{(1+n^2)}(z,z_n)z_n$.

It is easy to see that $-AL^{-1}$ generates a uniformly continuous semigroup $T(t)$, $t\geq 0$ and so $\max_{t\in[0,T]}\|T(t)\|$ is finite.

Let $h(t) = e^{2t}$, $t < 0$, then $l = \int_{-\infty}^{0} \frac{h(s)}{s} ds = \frac{1}{2}$ and define

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} (E|\theta|^2)^{\frac{1}{2}} ds \ .$$

Hence for $(t, \phi) \in [0, T] \times \mathcal{B}_h$, where $\phi(\theta)(x) = \phi(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$. But $z(t) = z(t, \cdot)$, that is $z(t)(x) = z(t, x)$. Define $f : [0, T] \times \mathcal{B}_h \longrightarrow L^2([0, \pi])$ and $\sigma : [0, T] \times \mathcal{B}_h \longrightarrow \mathcal{L}_2^0$ as follows:

$$\begin{aligned} f(t, \phi)x &= F(t, x(\cdot)), \\ \sigma(t, \phi)x &= G(t, x(\cdot)). \end{aligned}$$

With the choice of A, f and σ can be rewritten as the abstract form of system (2.1). Under the appropriate conditions on the functions f, σ and I_k as those in (H1)-(H3), system (3.6) has a mild solution on $(-\infty, T]$.

4. Controllability results

In this section, we treat the controllability of Sobolev-type impulsive stochastic fractional equations with nonlocal conditions using the argument of the previous section. More precisely we will consider the following problem:

$$\begin{cases} D_t^{\alpha} Lx(t) = Ax(t) + Bu(t) + f(t, x_t)) + \sigma(t, x_t)\dfrac{d\omega(t)}{dt}, & t \in J = [0, T], T > 0, t \neq t_k \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad k = 1, \dots, m \\ x(0) + g(x) = x_0 = \phi, \quad \phi \in B_h \ , \end{cases} \tag{4.1}$$

where $A, L,\ f, \sigma$ and I_k are as in Section 3, the nonlocal function $g : \mathcal{B}_h \longrightarrow \mathcal{H}$. The control function $u(\cdot)$ is given in $L^2(J, `\mathcal{U})$ a Banach space of admissible control functions for a separable Hilbert space $\mathcal{U}$. Finally B is a bounded linear operator from $\mathcal{U}$ to $\mathcal{H}$.

Definition 4.1. An $\mathcal{F}_t$-adapted stochastic process $x : (-\infty, T] \longrightarrow \mathcal{H}$ is said to be a mild solution of (4.1) if $x_0 = \phi(t)$ on $(-\infty, 0]$:

1. $x(t)$ is $\mathcal{B}_h$-valued and the restriction of $x(\cdot)$ to $(t_k, t_{k+1}]$, $k = 1, 2, \dots, m$ is continuous.
2. for each $t \in J$, $x(t)$ satisfies the following integral equation:

$$\begin{aligned} x(t) = T_L(t)L(\phi(0) - g(x)) \quad &+ \quad \int_0^t (t-s)^{\alpha-1} S_L(t-s) Bu(s) ds \\ &+ \quad \int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, x_s) ds \\ &+ \quad \int_0^t (t-s)^{\alpha-1} S_L(t-s) \sigma(s, x_s) d\omega(t) \\ &+ \quad \textstyle\sum_{0<t_k<t} T_L(t-t_k) I_k(x(t_k^-)), \end{aligned} \tag{4.2}$$

P - a.s for all $t \in J$.

Definition 4.2. The system (4.1) is said to be controllable on the interval $(-\infty, T]$ if for every initial value ϕ and every $x_1 \in \mathcal{H}$, there exists a control $u \in L^2(J,\mathcal{U})$, such that the mild solution $x(t)$ of system (4.1) satisfies $y(T) = x_1$.

Our main result in this section is the following.
We shall assume some additional hypotheses:

(H4) The linear operator W from $L^2(J,\mathcal{U})$ into $\mathcal{H}$ defined by

$$Wu = \int_0^T (T-s)^{\alpha-1} S_L(T-s)Bu(s)ds$$

has an induced inverse W^{-1} which takes values in $L^2(J,\mathcal{U})$ $kerW$, and there exist positive constants M_1, M_2 such that $\|B\|^2 = M_1$, $\|W^{-1}\|^2 = M_2$.

(H5) There exists $L_g > 0$ such that

$$E\|g(x) - g(y)\|^2_{\mathcal{H}} \leq L_g \|x - y\|^2_{\mathcal{B}_h}, \quad t \geq 0.$$

Theorem 4.3. *Assume that* $f(t, 0) = \sigma(t, 0) = g(0) = 0$, $\forall t \geq 0$. *Assume that hypotheses* $(H1) - (H3)$ *and* $(H4) - (H5)$ *hold. If*

$$r^* \geq 5M^2C_1^2(\sum_{k=1}^{m} L_k + C_2^2 L_g \chi^*) + \frac{5M^2C_1^2T^{2\alpha}}{\Gamma^2(\alpha)} \left[\frac{\chi^* L_f}{\alpha^2} + \frac{\chi^* L_\sigma}{T(2\alpha - 1)} + \frac{\xi M_1}{T(2\alpha - 1)} \right] \quad (4.3)$$

and

$$\Lambda = 4M^2C_1^2C_2^2L_g l + 4\frac{M^2C_1^2}{\Gamma^2(\alpha)}T^{2\alpha} \left[\frac{L_f l}{\alpha^2} \left(1 + \frac{3M_1M_2M^2C_1^2T^{2\alpha}}{\alpha^2\Gamma^2(\alpha)} \right) + \frac{L_\sigma l}{T(2\alpha - 1)} \left(1 + \frac{3M_1M_2M^2C_1^2T^{2\alpha}}{T(2\alpha - 1)\alpha^2\Gamma^2(\alpha)} \right) \right] < 1 \ , \quad (4.4)$$

then the system (4.1) *is controllable on* $(-\infty, T]$.

Proof. Using assumption (H4), for an arbitrary process $x(\cdot)$, define the control process

$$\begin{aligned} u_x(t) =& W^{-1} \Big\{ x_1 - T_L(t)L(\phi(0) - g(x)) - \int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, x_s) ds \\ & - \int_0^t (t-s)^{\alpha-1} S_L(t-s) \sigma(s, x_s) d\omega(t) \\ & - \sum_{0<t_k<t} T_L(t-t_k) I_k(x(t_k^-)) \Big\} (t). \end{aligned} \quad (4.5)$$

It shall now be shown that when using this control, the operator Ψ^* defined by

$$\Psi^* x(t) = \begin{cases} \phi(t), \quad \text{if} \quad t \in (-\infty, 0] \\ T_L(t)L(\phi(0) - g(x)) + \int_0^t (t-s)^{\alpha-1} S_L(t-s)Bu(s)ds \\ + \int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, x_s)ds + \int_0^t (t-s)^{\alpha-1} S_L(t-s)\sigma(s, x_s)d\omega(t) \\ + \sum_{0<t_k<t} T_L(t-t_k) I_k(x(t_k^-)) \quad \text{for all} \quad t \in J, \end{cases}$$

from $\mathcal{B}_T$ into itself for each $y \in \mathcal{B}_T$ has a fixed point. This fixed point is then a solution of equation (4.1).

For $\phi \in \mathcal{B}_h$, we define $\widehat{\phi}$ by

$$\widehat{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ T_L(t)L\phi(0), & t \in [0, +\infty[; \end{cases} \quad \text{then } \widehat{\phi} \in \mathcal{B}_T.$$

Let $x(t) = y(t) + \widehat{\phi}(t), -\infty < t < T$.
It is evident that y satisfies $y_0 = 0, t \in (-\infty, 0]$, and

$$\begin{aligned} y(t) = & - T_L(t)Lg(y + \widehat{\phi}) + \int_0^t (t-s)^{\alpha-1} S_L(t-s)Bu_{y+\widehat{\phi}}(s)ds \\ & + \int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, y_s + \widehat{\phi}_s)ds \\ & + \int_0^t (t-s)^{\alpha-1} S_L(t-s)\sigma(s, y_s + \widehat{\phi}_s)d\omega(t) \\ & + \sum_{0<t_k<t} T_\alpha(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)), \ t \in J, \end{aligned}$$

where $u_{y+\widehat{\phi}}$ is obtained from (4.5) by replacing x_t by $y_t + \widehat{\phi}_t$.

Set $\mathcal{B}_T^0 = \{z \in \mathcal{B}_T, \quad \text{such that} \quad z_0 = 0\}$ and for any $z \in \mathcal{B}_T^0$ we have

$$\|z\|_{\mathcal{B}_T^0} = \|z_0\|_{\mathcal{B}_h} + \sup_{t \in J}(E\|z(t)\|^2)^{\frac{1}{2}} = \sup_{t \in J}(E\|z(t)\|^2)^{\frac{1}{2}},$$

where $\|z_0\|_{\mathcal{B}_h} = 0$. Thus $(\mathcal{B}_T^0, \|.\|_{\mathcal{B}_T^0})$ is a Banach space.

Let the operator $\widehat{\Psi}^* : \mathcal{B}_T^0 \longrightarrow \mathcal{B}_T^0$ be defined by

$$\widehat{\Psi}^* y(t) = \begin{cases} 0, \quad t \in (-\infty, 0] \\ -T_L(t)Lg(y + \widehat{\phi}) + \int_0^t (t-s)^{\alpha-1} S_L(t-s)Bu_{y+\widehat{\phi}}(s)ds \\ + \int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, y_s + \widehat{\phi}_s)ds \\ + \int_0^t (t-s)^{\alpha-1} S_L(t-s)\sigma(s, y_s + \widehat{\phi}_s)d\omega(t) \\ + \sum_{0<t_k<t} T_L(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)), \ t \in J. \end{cases}$$

Set $\mathcal{B}_{r^*} = \left\{ y \in \mathcal{B}_T^0, \quad \|y\|^2_{\mathcal{B}_T^0} \leq r^*, \ r^* > 0 \right\}$. The set $\mathcal{B}_{r^*}$ is clearly a bounded closed convex set in $\mathcal{B}_T^0$ for each $r^* > 0$ and for each $y \in \mathcal{B}_{r^*}$. By Lemma 2.2 we have

$$\begin{aligned}
\|y_t + \widehat{\phi}_t\|^2_{\mathcal{B}_h} \leq \ & 2(\|y_t\|^2_{\mathcal{B}_h} + \|\widehat{\phi}_t\|^2_{\mathcal{B}_h}) \\
& \leq 4(l^2 \sup\nolimits_{s\in[0,t]} E\|y(s)\|^2_{\mathcal{H}} + \|y_0\|^2_{\mathcal{B}_h}) \\
& +4(l^2 \sup\nolimits_{s\in[0,t]} E\|\widehat{\phi}(s)\|^2_{\mathcal{H}} + \|\widehat{\phi}_0\|^2_{\mathcal{B}_h}) \\
& \leq 4\|\phi\|^2_{\mathcal{B}_h} + 4l^2(r^* + M^2C_1^2C_2^2E\|\phi(0)\|^2_{\mathcal{H}}) = \chi^* \ .
\end{aligned}$$

Now, let the two operators $\widehat{\Psi}_1^*$ and $\widehat{\Psi}_2^*$ defined as

$$\widehat{\Psi}_1^* y(t) = \begin{cases} 0 & t \in (-\infty, 0] \\ \displaystyle\sum_{0<t_k<t} T_L(t-t_k) I_k(y(t_k^-) + \widehat{\phi}(t_k^-)), & t \in [0,T] \end{cases} \tag{4.6}$$

and

$$(\widehat{\Psi}_2^* y)(t) = \begin{cases} 0 & t \in (-\infty, 0] \\ -T_L(t)Lg(y+\widehat{\phi}) + \displaystyle\int_0^t (t-s)^{\alpha-1} S_L(t-s) B u_{y+\widehat{\phi}}(s) ds & \\ + \displaystyle\int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, y_s + \widehat{\phi}_s) ds & \\ + \displaystyle\int_0^t (t-s)^{\alpha-1} S_L(t-s) \sigma(s, y_s + \widehat{\phi}_s) dW(t) & t \in [0,T] \ . \end{cases} \tag{4.7}$$

It is clear that

$$\widehat{\Psi}_1^* + \widehat{\Psi}_2^* = \Psi^* \ .$$

Then, the problem of finding a solution of (4.1) is reduced to finding a solution of the operator equation $y(t) = \widehat{\Psi}_1^*(y)(t) + \widehat{\Psi}_2^*(y)(t), t \in (-\infty, T]$. In order to use Theorem 2.8 we will verify that $\widehat{\Psi}_1^*$ is compact and continuous while $\widehat{\Psi}_2^*$ is a contraction operator.

For the sake of convenience, we divide the proof into several steps.

Step 1. We show that $\widehat{\Psi}_1^* y + \widehat{\Psi}_2^* y^* \in \mathcal{B}_{r^*}$, for $y, y^* \in \mathcal{B}_{r^*}$. For $t \in [0,T]$, we have

$$\begin{aligned}
\|(\widehat{\Psi}_1 y)(t) + (\widehat{\Psi}_2 y^*)(t)\|_{\mathcal{H}}^2 \le\ & 5 \sum_{0<t_k<t} \left\| T_L(t-t_k)\|^2 E\|I_k(y(t_k^-) + \widehat{\phi}(t_k^-)) \right\|_{\mathcal{H}}^2 \\
& + 5\|T_L(t)L\|^2 E\|g(y+\widehat{\phi})\|_{\mathcal{H}}^2 \\
& + 5E\left\| \int_0^t (t-s)^{\alpha-1} S_L(t-s) f(s, y_s^* + \widehat{\phi}_s) ds \right\|_{\mathcal{H}}^2 \\
& + 5E\left\| \int_0^t (t-s)^{\alpha-1} S_L(t-s) \sigma(s, y_s^* + \widehat{\phi}_s) d\omega(t) \right\|_{\mathcal{H}}^2 \\
& + 5 \int_0^t (t-s)^{\alpha-1} S_L(t-s) B u_{y^*+\widehat{\phi}}(s) ds \ .
\end{aligned}$$

Observe that

$$\begin{aligned}
E\|u_{y^*+\widehat{\phi}}\|^2 \le 6M_2 \Bigg\{ & E|x_1|_{\mathcal{H}}^2 + M^2 C_1^2 C_2^2 E|\phi(0)|_{\mathcal{H}}^2 + M^2 C_1^2 C_2^2 L_g \chi^* + M^2 C_1^2 \sum_{k=1}^m L_k \\
& + \frac{M^2 C_1^2 T^{2\alpha} \chi^*}{\Gamma^2(\alpha)} \left[\frac{L_f}{\alpha^2} + \frac{L_\sigma}{T(2\alpha-1)} \right] \Bigg\} = \xi \ .
\end{aligned}$$

Then

$$\begin{aligned}
\|(\widehat{\Psi}_1 y)(t) + (\widehat{\Psi}_2 y^*)(t)\|_{\mathcal{H}}^2 \le\ & 5M^2 C_1^2 \sum_{k=1}^m L_k + 5\|T_L(t)L\|^2 E\|g(y^*+\widehat{\phi})\|_{\mathcal{H}}^2 \\
& + 5 \int_0^t \|(t-s)^{\alpha-1} S_L(t-s)\|^2 E\|f(s, y_s^* + \widehat{\phi}_s)\|_{\mathcal{H}}^2 ds \\
& + 5 \int_0^t \|(t-s)^{\alpha-1} S_\alpha(t-s)\|^2 E\|\sigma(s, y_s^* + \widehat{\phi}_s)\|_{\mathcal{H}}^2 ds \\
& + 5 \int_0^t (t-\eta)^{\alpha-1} S_L(t-\eta) B u_{y^*+\widehat{\phi}}(\eta) d\eta \\
\le\ & 5M^2 C_1^2 \sum_{k=1}^m L_k + 5M^2 C_1^2 C_2^2 L_g \chi^* \\
& + \frac{5M^2 C_1^2 L_f \chi^*}{\Gamma^2(\alpha)} \frac{T^\alpha}{\alpha} \int_0^t (t-s)^{\alpha-1}\ ds \\
& + \frac{5M^2 C_1^2 L_\sigma \chi^*}{\Gamma^2(\alpha)} \int_0^t (t-s)^{2(\alpha-1)} ds \\
& + \frac{5M^2 C_1^2 M_1 \xi}{\Gamma^2(\alpha)} \int_0^t (t-\eta)^{2(\alpha-1)} d\eta
\end{aligned}$$

$$
\begin{aligned}
&\leq 5M^2C_1^2(\sum_{k=1}^{m} L_k + C_2^2 L_g \chi^*) \\
&\quad + \frac{5M^2C_1^2L_f\chi^*}{\Gamma^2(\alpha)}\frac{T^{2\alpha}}{\alpha^2} + \frac{5M^2C_1^2L_\sigma\chi^*}{\Gamma^2(\alpha)}\frac{T^{2\alpha-1}}{2\alpha-1} \\
&\quad + \frac{5M^2C_1^2M_1\xi}{\Gamma^2(\alpha)}\frac{T^{2\alpha-1}}{2\alpha-1} \\
&= 5M^2C_1^2(\sum_{k=1}^{m} L_k + C_2^2 L_g \chi^*) \\
&\quad + \frac{5M^2C_1^2T^{2\alpha}}{\Gamma^2(\alpha)}\left[\frac{\chi^* L_f}{\alpha^2} + \frac{\chi^* L_\sigma}{T(2\alpha-1)} + \frac{\xi M_1}{T(2\alpha-1)}\right].
\end{aligned}
$$

Then

$$
\begin{aligned}
\|(\Psi_1 y)(t) + (\Psi_2 y^*)(t)\|^2_{\mathcal{H}} &\leq 5M^2C_1^2(\sum_{k=1}^{m} L_k + C_2^2 L_g \chi^*) \\
&\quad + \frac{5M^2C_1^2T^{2\alpha}}{\Gamma^2(\alpha)}\left[\frac{\chi^* L_f}{\alpha^2} + \frac{\chi^* L_\sigma}{T(2\alpha-1)} + \frac{\xi M_1}{T(2\alpha-1)}\right] \leq r^*.
\end{aligned}
$$

Hence, we get $\widehat{\Psi}_1^* y + \widehat{\Psi}_2^* y^* \in \mathcal{B}_r^*$.

Step 2. As in Section 3, we can prove that the operators $\widehat{\Psi}_1^*$ is compact and continuous.

Step 3. $\widehat{\Psi}_2^*$ is a contraction mapping.

Let y ,$y^* \in \mathcal{B}_{r^*}$ and $t \in J$ we have

$$
\begin{aligned}
\|(\widehat{\Psi}_2^* y)(t) - (\widehat{\Psi}_2^* y^*)(t)\|^2_{\mathcal{H}} &\leq 4E\left\|T_L(t)L\left[g(y+\widehat{\phi}) - g(y^*+\widehat{\phi})\right]\right\|^2_{\mathcal{H}} \\
&\quad + 4E\left\|\int_0^t (t-s)^{\alpha-1}S_L(t-s)\left[f(s,y_s+\widehat{\phi}_s) - f(s,y_s^*+\widehat{\phi}_s)\right]ds\right\|^2_{\mathcal{H}} \\
&\quad + 4E\left\|\int_0^t (t-s)^{\alpha-1}S_L(t-s)\left[\sigma(s,y_s+\widehat{\phi}_s) - \sigma(s,y_s^*+\widehat{\phi}_s)\right]d\omega(s)\right\|^2_{\mathcal{H}} \\
&\quad + 4E\left\|\int_0^t (t-\eta)^{\alpha-1}S_L(t-\eta)B(u_{y+\widehat{\phi}} - u_{y^*+\widehat{\phi}})d\eta\right\|^2_{\mathcal{H}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq 4M^2C_1^2C_2^2E\|g(y+\widehat{\phi})-g(y^*+\widehat{\phi})\|^2_{\mathcal{H}}\\
&+4\int_0^t\|(t-s)^{\alpha-1}S_L(t-s)\|ds\int_0^t\|(t-s)^{\alpha-1}S_L(t-s)\|\\
&\times E\|f(s,y_s+\widehat{\phi}_s)-f(s,y_s^*+\widehat{\phi}_s)\|^2_{\mathcal{H}}ds\\
&+4\int_0^t\|(t-s)^{\alpha-1}S_L(t-s)\|^2E\|\sigma(s,y_s+\widehat{\phi}_s)-\sigma(s,y_s^*+\widehat{\phi}_s)\|^2_{L_2^0}ds\\
&+12M_1M_2\int_0^t\|(t-\eta)^{\alpha-1}S_L(T-\eta)\|d\eta\int_0^t\|(t-\eta)^{\alpha-1}S_L(T-\eta)\|\\
&\times\left[\int_0^t\|(t-s)^{\alpha-1}S_L(t-s)\|ds\int_0^t\|(t-s)^{\alpha-1}S_L(t-s)\|\right.\\
&\left.\times E\|f(s,y_s+\widehat{\phi}_s)-f(s,y_s^*+\widehat{\phi}_s)\|^2_{\mathbb{H}}ds\right]d\eta\\
&+12M_1M_2\int_0^t\|(t-\eta)^{\alpha-1}S_L(T-\eta)\|^2\left[\int_0^t\|(t-s)^{\alpha-1}S_L(t-s)\|^2\right.\\
&\left.\times\ E\|\sigma(s,y_s+\widehat{\phi}_s)-\sigma(s,y_s^*+\widehat{\phi}_s)\|^2_{\mathcal{H}}ds\right]d\eta\\
&+12M_1M_2M^2C_1^2C_2^2E\|g(y+\widehat{\phi})-g(y^*+\widehat{\phi})\|^2_{\mathcal{H}}\\
&\leq 4M^2C_1^2C_2^2L_g\|y-y^*\|^2_{\mathcal{B}_h}\\
&+\frac{4M^2C_1^2}{\Gamma^2(\alpha)}\int_0^t(t-s)^{\alpha-1}ds\int_0^t(t-s)^{\alpha-1}L_f\|y_s-y_s^*\|^2_{\mathcal{B}_h}ds\\
&+\frac{4M^2C_1^2}{\Gamma^2(\alpha)}\int_0^t(t-s)^{2(\alpha-1)}L_\sigma\|y_s-y_s^*\|^2_{\mathcal{B}_h}ds\\
&+\frac{12M_1M_2M^4C_1^4}{\Gamma^4(\alpha)}\int_0^t(t-\eta)^{\alpha-1}d\eta\int_0^t(t-\eta)^{\alpha-1}\\
&\times\left[\int_0^t(t-s)^{(\alpha-1)}ds\int_0^t(t-s)^{(\alpha-1)}L_f\|y_s-y_s^*\|^2_{\mathcal{B}_h}ds\right]d\eta\\
&+\frac{12M_1M_2M^4C_1^4}{\Gamma^4(\alpha)}\int_0^t(t-\eta)^{2(\alpha-1)}\left[\int_0^t(t-s)^{2(\alpha-1)}L_\sigma\|y_s-y_s^*\|^2_{\mathcal{B}_h}ds\right]d\eta\\
&+12M_1M_2M^2C_1^2C_2^2L_g\|y-y^*\|^2_{\mathcal{B}_h}\\
&\leq\left(4M^2C_1^2C_2^2L_gl+4\frac{M^2C_1^2}{\Gamma^2(\alpha)}T^{2\alpha}\left[\frac{L_fl}{\alpha^2}\left(1+\frac{3M_1M_2M^2C_1^2T^{2\alpha}}{\alpha^2\Gamma^2(\alpha)}\right)\right.\right.\\
&\left.\left.+\frac{L_\sigma l}{T(2\alpha-1)}\left(1+\frac{3M_1M_2M^2C_1^2T^{2\alpha}}{T(2\alpha-1)\alpha^2\Gamma^2(\alpha)}\right)\right]\right)\|y-y^*\|^2_{\mathcal{B}_T^0}\\
&=\Lambda\|y-y^*\|^2_{\mathcal{B}_T^0}\ .
\end{aligned}
$$

By the condition (3.3), we obtain that $\widehat{\Psi}_2^*$ is a contraction mapping. Hence, by Krasnoselskii's fixed point theorem we can conclude that the problem (4.1) has a mild solution on $(-\infty, T]$ and clearly, $x(T) = (\Psi^* x)(T)$, which implies that the system (4.1) is controllable on $(-\infty, T]$. This completes the proof of the theorem. □

Example 4.4. Now, we present an example to illustrate Theorem 4.3. Consider the fractional partial stochastic differential equation in the following form

$$\begin{cases} D_t^\alpha[z(t,x) - z_{xx}(t,x)] = z_{xx}(t,x) + \mu(x,t) + F(t, z(t-R,x)) \\ +G(t, z(t-R,x))\dfrac{d\omega(t)}{dt}, \ x \in= [0,\pi], R > 0, t \neq t_k \\ I_k(z(t_k^-, x)) = z(t_k^+, x) - z(t_k^-, x), \quad k = 1, \dots, n \\ z(t,x) = \phi(t,x), \quad t \in (-\infty, 0], \ x \in [0,\pi]. \\ x(0,x) + \int_0^\pi H(x,y) z(t,y) dy = \phi(t,x), \quad t \in (-\infty, 0]. \end{cases} \tag{4.8}$$

where $\omega(t)$ is a standard cylindrical Wiener process in $\mathcal{H}$ defined on a stochastic space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$; D_t^α is the Caputo fractional derivative of order $0 < \alpha < 1$; $0 < t_1 < t_2 < \ldots < t_n < T$ are prefixed numbers. Let $h(t) = e^{2t}$, $t < 0$, then $l = \int_{-\infty}^0 \frac{h(s)}{s} ds = \frac{1}{2}$ and define

$$\|\phi\|_{\mathcal{B}_h} = \int_{-\infty}^0 h(s) \sup_{s \leq \theta \leq 0} (E|\theta|^2)^{\frac{1}{2}} ds.$$

Let $\mathcal{K} = \mathcal{H} = L^2([0,\pi])$ with the norm $\|\cdot\|$. Define the operators $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$, $L : D(L) \subset \mathcal{H} \longrightarrow \mathcal{H}$, by $Az = -z''$ and $Lz = z - z''$, where each domain $D(A)$ and $D(L)$ is given by $\{z \in \mathcal{H}, z, z'$ are absolutely continuous , $z'' \in \mathcal{H}$ and $z(0) = z(\pi) = 0\}$.

Further, A and L can be $Az = \sum_{n=1}^{\infty} n^2 (z, z_n) z_n$, $z \in D(A)$, $Lz = \sum_{n=1}^{\infty} (1 + n^2)(z, z_n) z_n$, $z \in D(L)$, where $z_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx), n = 1, 2, \ldots,$ is the orthogonal set of vectors of A. Also, for $z \in \mathcal{H}$ $L^{-1}z = \sum_{n=1}^{\infty} \frac{1}{(1+n^2)} (z, z_n) z_n$, $AL^{-1}z = \sum_{n=1}^{\infty} \frac{n}{(1+n^2)} (z, z_n) z_n$, $T(t)z = \sum_{n=1}^{\infty} exp \frac{n^2 t}{(1+n^2)} (z, z_n) z_n$.
It is easy to see that $-AL^{-1}$ generates a uniformly continuous semigroup $T(t)$, $t \geq 0$ and so $\max_{t \in [0,T]} \|T(t)\|$ is finite.

Hence for $(t, \phi) \in [0, T] \times \mathcal{B}_h$, where $\phi(\theta)(x) = \phi(\theta, x)$, $(\theta, x) \in (-\infty, 0] \times [0, \pi]$. But $z(t) = z(t, \cdot)$, that is $z(t)(x) = z(t, x)$. Define $f : [0,T] \times \mathcal{B}_h \longrightarrow L^2([0,\pi])$, $\sigma : [0,T] \times \mathcal{B}_h \longrightarrow \mathcal{L}_2^0$. The bounded linear operator $B : \mathcal{U} \longrightarrow \mathcal{H}$ is defined by $Bu(t)(x) = \mu(t,x)$, $0 \leq x \leq \pi$, $u \in \mathcal{U}$; as follows:

$$g(z)(x) = \int_0^\pi H(x,y)z(t,y)dy \ ,$$

$$f(t,\phi)x = F(t,x(\cdot)) \ ,$$

$$\sigma(t,\phi)x = G(t,x(\cdot)) \ .$$

With the choice of A, f, g and σ can be rewritten as the abstract form of system (4.1). Under the appropriate conditions on the functions f, σ, g and I_k as those in (H1)-(H3) and (H4)-(H5), system (4.8) is controllable on $(-\infty, T]$.

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Oscillation of Second Order Difference Equation with a Sub-linear Neutral Term

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Abstract: This paper deals with the oscillation of a certain class of second order difference equations with a sub-linear neutral term. Using some inequalities and Riccati type transformation, four new oscillation criteria are obtained. Examples are included to illustrate the main results.

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Keywords and Phrases: Difference equations; Oscillation; Sub-linear neutral term; Second order.

1. Introduction

In this paper, we are concerned with the oscillatory behavior of the nonlinear difference equation with a sub-linear neutral term

$$\Delta(a_n\Delta(x_n + p_n x^{\alpha}_{n-k})) + q_n x^{\beta}_{n+1-l} = 0,\ n \geq n_0, \tag{1.1}$$

where n_0 is a nonnegative integer, subject to the following conditions:

(H_1) $0 < \alpha \leq 1$ and β are ratios of odd positive integers;

(H_2) $\{a_n\}$, $\{p_n\}$, and $\{q_n\}$ are positive real sequences for all $n \geq n_0$;

(H_3) k is a positive integer, and l is a nonnegative integer.

Let $\theta = \max\{k, l\}$. By a *solution* of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n \geq n_0 - \theta$ that satisfies equation (1.1) for all $n \geq n_0$. A solution of equation (1.1) is called *oscillatory* if its terms are neither eventually positive nor eventually negative, and *nonoscillatory* otherwise.

In the last few years there has been a great interest in investigating the oscillatory and asymptotic behavior of neutral type difference equations, see [1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the references cited therein.

In [4], Lin considered the equation of the form

$$\Delta(x_n - p_n x_{n-k}^{\alpha}) + q_n x_{n-l}^{\beta} = 0,\ n \geq n_0, \tag{1.2}$$

and studied its oscillatory behavior. In [5], Thandapani et al. investigated the oscillation of all solutions of the equation

$$\Delta(a_n \Delta(x_n - p x_{n-k}^{\alpha})) + q_n x_{n+1-l}^{\beta} = 0,\ n \geq n_0, \tag{1.3}$$

where $p > 0$ is a real number, k and l are positive integers, $0 < \alpha \leq 1$ and β are ratios of odd positive integers, and $\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty$.

A special case of the equation studied by Yildiz and Ogunmez [11] has the form

$$\Delta^2(x_n + p_n x_{n-k}^{\alpha}) + q_n x_{n-l}^{\beta} = 0, \tag{1.4}$$

where $\{p_n\}$ is a real sequence, $\{q_n\}$ is a nonnegative real sequence, and $\alpha > 1$ and $\beta > 0$ are again ratios of odd positive integers. They too discussed the oscillatory behavior of solutions.

In [6], Thandapani et al. considered equation (1.3), and obtained criteria for the oscillation of solutions provided $\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty$.

In this paper, we obtain sufficient conditions for the oscillation of all solutions of equation (1.1) in the two cases

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} = \infty \tag{1.5}$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n} < \infty. \tag{1.6}$$

Our technique of proof makes use of some inequalities and Riccati type transformations. The results we obtain here are new and generalize those reported in [4, 5, 6, 11, 12]. Examples are provided to illustrate the main results.

2. Oscillation results

In this section, we obtain sufficient conditions for the oscillation of all solutions of equation (1.1). We set

$$z_n = x_n + p_n x_{n-k}^{\alpha}.$$

Due to the form of our equation, we only need to give proofs for the case of eventually positive nonoscillatory solutions since the proofs for eventually negative solutions would be similar.

We begin with the following two lemmas given in [7].

Lemma 2.1. *Assume that $\beta \geq 1$ and $a,\ b \in [0, \infty)$. Then*

$$a^{\beta} + b^{\beta} \geq \frac{1}{2^{\beta-1}}(a+b)^{\beta}.$$

Lemma 2.2. *Assume that* $0 < \beta \le 1$ *and* $a, b \in [0, \infty)$. *Then*

$$a^\beta + b^\beta \ge (a+b)^\beta.$$

The next lemma can be found in [3, Theorem 41, p. 39].

Lemma 2.3. *Assume that* $a > 0$, $b > 0$, *and* $0 < \beta \le 1$. *Then*

$$a^\beta - b^\beta \le \beta b^{\beta-1}(a-b).$$

Here is our first oscillation result.

Theorem 2.4. *Assume that* (H_1)–(H_3) *and* (1.5) *hold. If* $\beta \ge 1$ *and there exists a positive nondecreasing real sequence* $\{\rho_n\}$ *such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left(\left[\frac{(1-\alpha p_{s+1-l})^\beta}{2^{\beta-1}} - \frac{(1-\alpha)^\beta p_{s+1-l}^\beta}{M^\beta} \right] \rho_s q_s - \frac{a_{s-l}(\Delta\rho_s)^2}{4\beta M^{\beta-1}\rho_s} \right) = \infty \quad (2.1)$$

holds for all constants $M > 0$, *then every solution of equation* (1.1) *is oscillatory.*

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution $\{x_n\}$, say $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \ge n_1$ for some $n_1 \ge n_0$. From equation (1.1), we have

$$\Delta(a_n \Delta z_n) = -q_n x_{n+1-l}^\beta < 0, \; n \ge n_1. \quad (2.2)$$

In view of condition (1.5), it is easy to see that $\Delta z_n > 0$ for all $n \ge n_1$. Now, it follows from the definition z_n, and using Lemma 2.3, we have

$$\begin{aligned} x_n &= z_n - p_n x_{n-k}^\alpha \ge z_n - p_n(z_n^\alpha - 1) - p_n \\ &\ge z_n - \alpha p_n(z_n - 1) - p_n \\ &= (1-\alpha p_n)z_n - (1-\alpha)p_n \end{aligned}$$

or

$$(x_{n+1-l} + (1-\alpha)p_{n+1-l})^\beta \ge (1-\alpha p_{n+1-l})^\beta z_{n+1-l}^\beta, \; n \ge n_1.$$

Using Lemma 2.1, in the last inequality, we obtain

$$x_{n+1-l}^\beta \ge \frac{1}{2^{\beta-1}}(1-\alpha p_{n+1-l})^\beta z_{n+1-l}^\beta - (1-\alpha)^\beta p_{n+1-l}^\beta, \; n \ge n_1. \quad (2.3)$$

From (2.2) and (2.3), we have

$$\Delta(a_n \Delta z_n) \le \frac{-(1-\alpha p_{n+1-l})^\beta}{2^{\beta-1}} q_n z_{n+1-l}^\beta + (1-\alpha)^\beta q_n p_{n+1-l}^\beta, \; n \ge n_1. \quad (2.4)$$

Define

$$w_n = \frac{\rho_n a_n \Delta z_n}{z_{n-l}^\beta}, \; n \ge n_1. \quad (2.5)$$

Then, $w_n > 0$ for all $n \geq n_1$, and

$$\Delta w_n = \frac{\rho_n \Delta(a_n \Delta z_n)}{z^\beta_{n+1-l}} + \frac{(\Delta\rho_n) a_{n+1} \Delta z_{n+1}}{z^\beta_{n+1-l}} - \frac{\rho_n a_n \Delta z_n}{z^\beta_{n+1-l} z^\beta_{n-l}} \Delta(z^\beta_{n-l}). \tag{2.6}$$

By the Mean Value Theorem

$$z^\beta_{n+1-l} - z^\beta_{n-l} \geq \begin{cases} \beta z^\beta_{n-l} \Delta z_{n-l}, \ if \ \beta \geq 1, \\ \beta z^{\beta-1}_{n+1-l} \Delta z_{n-l}, \ if \ \beta < 1. \end{cases} \tag{2.7}$$

Combining (2.7) with (2.6) and then using the facts that $a_n \Delta z_n$ is positive and decreasing and z_n is increasing, we have

$$\begin{aligned} \Delta w_n \ &\leq \ \frac{-(1-\alpha p_{n+1-l})^\beta}{2^{\beta-1}} \rho_n q_n + \frac{\rho_n (1-\alpha)^\beta}{M^\beta} p^\beta_{n+1-l} \rho_n q_n \\ &\quad + \frac{\Delta \rho_n w_{n+1}}{\rho_{n+1}} - \beta M^{\beta-1} \frac{\rho_n}{\rho^2_{n+1} a_{n-l}} w^2_{n+1}, \ n \geq n_1, \end{aligned} \tag{2.8}$$

where we have used the fact that $z_n \geq M$ for some $M > 0$ and all $n \geq n_1$. Completing the square on the last two terms on the right, we obtain

$$\Delta w_n \ \leq \ -\left[\frac{(1-\alpha p_{n+1-l})^\beta}{2^{\beta-1}} - \frac{(1-\alpha)^\beta}{M^\beta} p^\beta_{n+1-l}\right] \rho_n q_n + \frac{a_{n-l}(\Delta\rho_n)^2}{4\beta M^{\beta-1} \rho_n}, \ n \geq n_1.$$

Summing the last inequality from n_1 to n yields

$$\sum_{s=n_1}^{n} \left(\left[\frac{(1-\alpha p_{s+1-l})^\beta}{2^{\beta-1}} - \frac{(1-\alpha)^\beta}{M^\beta} p^\beta_{s+1-l}\right] \rho_s q_s - \frac{a_{s-l}(\Delta\rho_s)^2}{4\beta M^{\beta-1} \rho_s} \right) \leq w_{n_1},$$

which contradicts (2.1) and completes the proof of the theorem. □

The proof of the following theorem is similar to that of Theorem 2.4 only using Lemma 2.2 instead of Lemma 2.1. We omit the details.

Theorem 2.5. *Assume that* (H_1)*–*(H_3) *and* (1.5) *hold. If* $0 < \beta < 1$ *and there exists a positive nondecreasing real sequence* $\{\rho_n\}$ *such that*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} \left(\left[(1-\alpha p_{s+1-l})^\beta - \frac{(1-\alpha)^\beta}{M^\beta} p^\beta_{s+1-l}\right] \rho_s q_s - \frac{a_{s-l}(\Delta\rho_s)^2}{4\beta M^{\beta-1} \rho_s} \right) = \infty \tag{2.9}$$

holds for all constants $M > 0$*, then every solution of equation* (1.1) *is oscillatory.*

Our next two theorems are for the case where (1.6) holds in place of (1.5). We let

$$A_n = \sum_{s=n}^{\infty} \frac{1}{a_s}.$$

We will also need the condition

$$1 - \alpha p_n \frac{A_{n-k}}{A_n} > 0 \text{ for all } n \geq n_0. \tag{2.10}$$

Theorem 2.6. *Let $\beta \geq 1$ and (H_1)–(H_3), (1.6), and (2.10) hold. Assume that there exists a positive nondecreasing real sequence $\{\rho_n\}$ such that (2.1) holds for all constants $M > 0$. If*

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left(A_{s+1}^{\beta} \left[\left(1 - \alpha p_{s+1-l} \frac{A_{s+1-l-k}}{A_{s+1-l}}\right)^{\beta} \frac{1}{2^{\beta-1}} - \frac{(1-\alpha)^{\beta} p_{s+1-l}^{\beta}}{D^{\beta} A_{s+1}^{\beta}} \right] q_s - \frac{\beta A_s^{\beta-1}}{4D^{\beta-1} a_s A_{s+1}^{\beta}} \right) = \infty \qquad (2.11)$$

holds for every constant $D > 0$, then every solution of equation (1.1) *is oscillatory.*

Proof. Assume to the contrary that equation (1.1) has an eventually positive solution such that $x_n > 0$, $x_{n-k} > 0$, and $x_{n-l} > 0$ for all $n \geq n_1 \geq n_0$. From (1.1), we have that (2.2) holds. We then have that either $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If $\Delta z_n > 0$ holds, then we can proceed as in the proof of Theorem 2.4 and again obtain a contradiction to (2.1).

Now assume that $\Delta z_n < 0$ for all $n \geq n_1$. Define

$$u_n = \frac{a_n \Delta z_n}{z_n^{\beta}}, \ n \geq n_1. \qquad (2.12)$$

Then $u_n < 0$ for all $n \geq n_1$ and from (2.2), we have

$$\Delta z_s \leq \frac{a_n \Delta z_n}{a_s}, \ s \geq n.$$

Summing the last inequality from n to j, we obtain

$$z_{j+1} - z_n \leq a_n \Delta z_n \sum_{s=n}^{j} \frac{1}{a_s};$$

and then letting $j \to \infty$ gives

$$\frac{a_n \Delta z_n A_n}{z_n} \geq -1, \ n \geq n_1. \qquad (2.13)$$

Thus,

$$\frac{-a_n \Delta z_n (-a_n \Delta z_n)^{\beta-1} A_n^{\beta}}{z_n^{\beta}} \leq 1$$

for $n \geq n_1$. Since $-a_n \Delta z_n > 0$ and (2.2) and (2.12) hold, we have

$$-\frac{1}{L^{\beta-1}} \leq u_n A_n^{\beta} \leq 0, \qquad (2.14)$$

where $L = -a_{n_1} \Delta z_{n_1}$. On the other hand, from (2.13),

$$\Delta \left(\frac{z_n}{A_n} \right) \geq 0, \ n \geq n_1. \qquad (2.15)$$

From the definition of z_n, (2.15), and Lemma 2.3, we have

$$\begin{aligned} x_n &= z_n - p_n x_{n-k}^{\alpha} \geq z_n - p_n(z_{n-k}^{\alpha} - 1) - p_n \\ &\geq z_n - \alpha p_n(z_{n-k} - 1) - p_n \\ &\geq \left(1 - \alpha p_n \frac{A_{n-k}}{A_n}\right) z_n + (\alpha - 1)p_n, \end{aligned}$$

or

$$(x_{n+1-l} + (1-\alpha)p_{n+1-l})^{\beta} \geq \left(1 - \alpha p_{n+1-l} \frac{A_{n+1-l-k}}{A_{n+1-l}}\right)^{\beta} z_{n+1-l}^{\beta}.$$

Using Lemma 2.1, in the last inequality, we obtain

$$x_{n+1-l}^{\beta} \geq \frac{1}{2^{\beta-1}} \left(1 - \alpha p_{n+1-l} \frac{A_{n+1-l-k}}{A_{n+1-l}}\right)^{\beta} z_{n+1-l}^{\beta} - (1-\alpha)^{\beta} p_{n+1-l}^{\beta}. \quad (2.16)$$

From (2.2) and (2.16), we have

$$\Delta(a_n \Delta z_n) \leq -\frac{q_n}{2^{\beta-1}} \left(1 - \alpha p_{n+1-l} \frac{A_{n+1-l-k}}{A_{n+1-l}}\right)^{\beta} z_{n+1-l}^{\beta} + q_n(1-\alpha)^{\beta} p_{n+1-l}^{\beta}. \quad (2.17)$$

From (2.12),

$$\Delta u_n = \frac{\Delta(a_n \Delta z_n)}{z_{n+1}^{\beta}} - \frac{a_n \Delta z_n}{z_n^{\beta} z_{n+1}^{\beta}} \Delta z_n^{\beta}, \; n \geq n_1. \quad (2.18)$$

By the Mean Value Theorem,

$$z_{n+1}^{\beta} - z_n^{\beta} \leq \begin{cases} \beta z_{n+1}^{\beta-1} \Delta z_n, \; if \; \beta \geq 1, \\ \beta z_n^{\beta-1} \Delta z_n, \; if \; 0 < \beta < 1, \end{cases} \quad (2.19)$$

so combining (2.19) and (2.18) and using the fact that $\Delta z_n < 0$ gives

$$\Delta u_n \leq \frac{\Delta(a_n \Delta z_n)}{z_{n+1}^{\beta}} - \beta \frac{u_n^2}{a_n} z_n^{\beta-1}. \quad (2.20)$$

Since z_n/A_n is increasing, there is a constant $D > 0$ such that $z_n/A_n \geq D$ for $n \geq n_1$. Using this together with (2.15) and (2.17) in (2.20), we obtain

$$\Delta u_n \leq \frac{-q_n}{2^{\beta-1}} \left(1 - \alpha p_{n+1-l} \frac{A_{n+1-l-k}}{A_{n+1-l}}\right)^{\beta} + \frac{q_n(1-\alpha)^{\beta}}{D^{\beta} A_{n+1}^{\beta}} p_{n+1-l}^{\beta} - \beta D^{\beta-1} A_n^{\beta-1} \frac{u_n^2}{a_n}. \quad (2.21)$$

Multiplying (2.21) by A_{n+1}^{β} and then summing the resulting inequality from n_1 to $n-1$, we see that

$$A_n^{\beta} u_n - A_{n_1}^{\beta} u_{n_1} + \sum_{s=n_1}^{n-1} A_{s+1}^{\beta} \left[\left(1 - \alpha p_{s+1-l} \frac{A_{s+1-l-k}}{A_{s+1-l}}\right)^{\beta} \frac{1}{2^{\beta-1}} - \frac{(1-\alpha)^{\beta}}{D^{\beta} A_{s+1}^{\beta}} p_{s+1-l}^{\beta} \right] q_s$$
$$+ \sum_{s=n_1}^{n-1} \frac{\beta A_s^{\beta-1} u_s}{a_s} + \sum_{s=n_1}^{n-1} \beta D^{\beta-1} A_s^{\beta-1} A_{s+1}^{\beta} \frac{u_s^2}{a_s} \leq 0,$$

which upon completing the square on the last two terms yields

$$\sum_{s=n_1}^{n-1}\left(A_{s+1}^{\beta}\left[\left(1-\alpha p_{s+1-l}\frac{A_{s+1-l-k}}{A_{s+1-l}}\right)^{\beta}\frac{1}{2^{\beta-1}}-\frac{(1-\alpha)^{\beta}}{D^{\beta}A_{s+1}^{\beta}}p_{s+1-l}^{\beta}\right]q_s - \frac{\beta A_s^{\beta-1}}{4D^{\beta-1}a_sA_{s+1}^{\beta}}\right)\le\frac{1}{L^{\beta-1}}+A_{n_1}u_{n_1}$$

in view of (2.14). This contradicts (2.11), and completes the proof of the theorem. □

The proof of the following theorem is similar to that of Theorem 2.6 using Lemma 2.2 instead of Lemma 2.1. We again omit the details.

Theorem 2.7. *Let* $0<\beta<1$ *and* (H_1)*–*(H_3)*,* (1.6)*, and* (2.10) *hold. Assume that there exists a positive nondecreasing real sequence* $\{\rho_n\}$ *such that* (2.9) *holds for all constants* $M>0$*. If*

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left(A_{s+1}^{\beta}\left[\left(1-\alpha p_{s+1-l}\frac{A_{s+1-l-k}}{A_{s+1-l}}\right)^{\beta}-\frac{(1-\alpha)^{\beta}p_{s+1-l}^{\beta}}{D^{\beta}A_{s+1}^{\beta}}\right]q_s - \frac{\beta A_s^{\beta-1}}{4D^{\beta-1}a_sA_{s+1}^{\beta}}\right)=\infty \quad (2.22)$$

holds for all constants $D>0$*, then every solution of equation* (1.1) *is oscillatory.*

3. Examples

In this section, we present two examples to illustrate our main results.

Example 3.1. Consider the neutral difference equation

$$\Delta\left((n+1)\Delta\left(x_n+\frac{1}{n}x_{n-2}^{1/3}\right)\right)+\left(4n+10+\frac{2n+1}{n(n+1)}\right)x_{n-3}^{3}=0,\ n\ge 1. \quad (3.1)$$

Here $a_n=(n+1)$, $p_n=\frac{1}{n}$, $q_n=4n+10+\frac{2n+1}{n(n+1)}$, $\alpha=\frac{1}{3}$, $\beta=3$, $k=2$, and $l=4$. By taking $\rho_n=1$, we see that all conditions of Theorem 2.4 are satisfied and hence every solution of equation (3.1) is oscillatory. In fact $\{x_n\}=\{(-1)^{3n}\}$ is one such oscillatory solution of equation (3.1).

Example 3.2. Consider the neutral difference equation

$$\Delta\left((n+1)(n+2)\Delta\left(x_n+\frac{1}{n(n+1)}x_{n-1}^{1/3}\right)\right)+\left(4(n+2)^2-\frac{2(2n^2+4n+1)}{n(n+1)}\right)x_{n-1}^{3}=0,\ n\ge 1. \quad (3.2)$$

Here $a_n = (n+1)(n+2)$, $p_n = \frac{1}{n(n+1)}$, $q_n = 4(n+2)^2 - \frac{2(2n^2+4n+1)}{n(n+1)}$, $\alpha = \frac{1}{3}$, $\beta = 3$, $k = 1$, and $l = 2$. Simple calculation shows that $A_n = \frac{1}{n+1}$ and $1 - \alpha p_n \frac{A_{n-k}}{A_n} = 1 - \frac{1}{3n^2} > 0$. The conditions (2.1) and (2.11) are also satisfied with $\rho_n = 1$. Therefore, by Theorem 2.6, every solution of equation (3.2) is oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such oscillatory solution of equation (3.2).

We conclude this paper with the following remark.

Remark 3.3. Condition (2.10) is somewhat restrictive. It implies that we must have $\{p_n\} \to 0$ as $n \to \infty$. It would be good to see a result that did not need this added condition. Note also that it can be seen from the proof of Theorem 2.6 that (2.10) is not needed if $\alpha = 1$.

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Measures of Noncompactness in a Banach Algebra and Their Applications

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Abstract: In this paper we study the existence of solutions of a nonlinear quadratic integral equation of fractional order. This equation is considered in the Banach space of real functions defined, continuous and bounded on the real half axis. Additionally, using the technique of measures of noncompactness we obtain some characterization of considered integral equation. We provide also an example illustrating the applicability of our approach.

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Keywords and Phrases: Banach algebra; Measure of noncompactness; Integral equation of fractional order; Fixed point theorem; Asymptotic stability; Ultimate monotonicity.

1. Introduction

Nonlinear functional-integral equations are often applicable in medicine, engineering, mathematical physics, radiative transfer, kinetic theory of gases and so on (cf. [3, 13, 14, 15, 16, 17, 20, 21], for example).

The purpose of the paper is to study the solvability of nonlinear quadratic integral equation of fractional order in the Banach algebra $BC(\mathbb{R}_+)$ consisting of real, continuous and bounded functions defined on an unbounded interval. The equation considered in this paper can be written as

$$x(t) = (U_1x)(t)(U_2x)(t),$$

where

$$(U_ix)(t) = m_i(t) + f_i(t, x(t)) \int_0^t \frac{v_i(t, s, x(s))}{(t-s)^{\alpha_i}} ds$$

for $t \in \mathbb{R}_+$, $\alpha_i \in (0, 1)$, $i = 1, 2$. Moreover, m_i, f_i, v_i are functions satisfying certain conditions for $i = 1, 2$.

Notice that differential and integral equations of fractional order create an important and significant branch of nonlinear analysis and the theory of integral equations (cf. [7, 8, 10, 16]).

Functional integral equations considered in Banach algebras have rather complicated form and the study of such equations requires of the use of sophisticated tools (cf. [6, 7, 11, 12, 18, 19]). We will use the technique associated with measures of noncompactness and some fixed point theorems [5]. The so-called condition (m) (introduced in [7]) related to the operation of multiplication in the algebra will be crucial in our considerations.

Measure of noncompactness used here allows us not only to obtain the existence of solutions of functional integral equation but also to characterize those solutions in terms of asymptotic stability and ultimate monotonicity.

The paper is a corrected and improved version of [4].

2. Notation, definitions and auxiliary results

In this section we collect some basic definitions and facts which will be used further on. At the beginning we introduce some notation.

Denote by $\mathbb{R}$ the set of real numbers and put $\mathbb{R}_+ = [0, \infty)$. Let $(E, \|\cdot\|)$ be a Banach space with zero element θ. Then by $B(x, r)$ we denote the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E, we use $\overline{X}$ and $\mathrm{Conv}X$ to denote the closure and convex closure of X, respectively. Apart from this the symbol $\mathrm{diam}X$ will denote the diameter of a bounded set X while $\|X\|$ denotes the norm of X i.e., $\|X\| = \sup\{\|x\| : x \in X\}$.

Next, let us denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting all relatively compact sets. We use the following definition of the measure of noncompactness given in [5].

Definition 2.1. A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ will be called *a measure of noncompactness* in E if it satisfies the following conditions:

1° The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.

2° $X \subset Y \Rightarrow \mu(X) \le \mu(Y)$.

3° $\mu(\overline{X}) = \mu(\mathrm{Conv}X) = \mu(X)$.

4° $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.

5° If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, ...$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker \mu$ described in 1° is said to be *the kernel of the measure of noncompactness* μ.

It is easy to show that the set X_∞ from the axiom 5° is a member of the family $\ker \mu$. Indeed, from the inequality $\mu(X_\infty) \le \mu(X_n)$ being satisfied for all $n = 1, 2, ...$

we derive that $\mu(X_\infty) = 0$ which means that $X_\infty \in \ker \mu$. This fact will play a key role in our further considerations.

In the sequel we will usually assume that the space E has the structure of Banach algebra. Then we write xy in order to denote the product of elements $x, y \in E$. Similarly, we will write XY to denote the product of subsets X, Y of E i.e., $XY = \{xy : x \in X,\, y \in Y\}$.

Now, we recall a useful concept (see [7]).

Definition 2.2. We say that the measure of noncompactness μ defined on the Banach algebra E satisfies *condition* (m) if for arbitrary sets $X, Y \in \mathfrak{M}_E$ the following inequality is satisfied:

$$\mu(XY) \leq \|X\|\mu(Y) + \|Y\|\mu(X).$$

It turns out that the above defined condition (m) is very convenient in considerations connected with the use of the technique of measures of noncompactness in Banach algebras.

For our purposes we will need a fixed point theorem for operators acting in a Banach algebra and satisfying some conditions expressed with help of a measure of noncompactness. To this end we first recall a concept parallel to the concept of Lipschitz continuity (cf. [5]).

Definition 2.3. Let Ω be a nonempty subset of a Banach space E and let $F : \Omega \to E$ be a continuous operator which transforms bounded subsets of Ω onto bounded ones. We say that F satisfies *the Darbo condition* with a constant k with respect to a measure of noncompactness μ if $\mu(FX) \leq k\mu(X)$ for each $X \in \mathfrak{M}_E$ such that $X \subset \Omega$. If $k < 1$, then F is called a *contraction* with respect to μ.

Now, assume that E is a Banach algebra and μ is a measure of noncompactness on E satisfying condition (m). Then we have the following theorem announced above [7].

Theorem 2.4. *Assume that Ω is nonempty, bounded, closed and convex subset of the Banach algebra E, and the operators P and T transform continuously the set Ω into E in such a way that $P(\Omega)$ and $T(\Omega)$ are bounded. Moreover, we assume that the operator $F = P \cdot T$ transforms Ω into itself. If the operators P and T satisfy on the set Ω the Darbo condition with respect to the measure of noncompactness μ with the constants k_1 and k_2 respectively, then the operator F satisfies on Ω the Darbo condition with the constant*

$$\|P(\Omega)\|k_2 + \|T(\Omega)\|k_1.$$

Particularly, if $\|P(\Omega)\|k_2 + \|T(\Omega)\|k_1 < 1$, then F is a contraction with respect to the measure of noncompactness μ and has at least one fixed point in the set Ω.

Remark 2.5. It can be shown [5] that the set $\mathrm{Fix}F$ of all fixed points of the operator F on the set Ω is a member of the kernel $\ker \mu$.

3. Some measure of noncompactness in the Banach algebra $BC(\mathbb{R}_+)$

In this section we present some measures of noncompactness in the Banach algebra $BC(\mathbb{R}_+)$ consisting of all real functions defined, continuous and bounded on the half axis R_+. The algebra $BC(\mathbb{R}_+)$ is endowed with the usual supremum norm

$$\|x\| \,=\, \sup\{|x(t)| : t \in \mathbb{R}_+\}$$

for $x \in BC(\mathbb{R}_+)$. Obviously, the multiplication in $BC(\mathbb{R}_+)$ is understood as the usual product of real functions. Let us mention that measures of noncompactness which we intend to present here, were considered in details in [7].

In what follows let us assume that X is an arbitrarily fixed nonempty and bounded subset of the Banach algebra $BC(\mathbb{R}_+)$ i.e., $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$. Choose arbitrary $\varepsilon > 0$ and $T > 0$. For $x \in X$ denote by $\omega^T(x, \varepsilon)$ *the modulus of continuity* of the function x on the interval $[0, T]$ i.e.,

$$\omega^T(x, \varepsilon) \,=\, \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \varepsilon\}.$$

Next, let us put:

$$\omega^T(X, \varepsilon) \,=\, \sup\{\omega^T(x, \varepsilon) : x \in X\},$$
$$\omega_0^T(X) \,=\, \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon),$$
$$\omega_0^\infty(X) \,=\, \lim_{T \to \infty} \omega_0^T(X).$$

In our considerations we will also need another set quantities. In order to define this quantities, let us fix $t \in \mathbb{R}_+$ and denote by $X(t)$ the cross-section of the set X at the point t i.e., $X(t) = \{x(t) : x \in X\}$. Denote by $\operatorname{diam} X(t)$ the diameter of $X(t)$. Further, for a fixed $T > 0$ and $x \in X$ denote by $d_T(x)$ the so-called *modulus of decrease* of the function x on the interval $[T, \infty)$, defined by the formula

$$d_T(x) \,=\, \sup\{|x(t) - x(s)| - [x(t) - x(s)] : T \leq s < t\}.$$

Further, let us put

$$d_T(X) \,=\, \sup\{d_T(x) : x \in X\},$$
$$d_\infty(X) \,=\, \lim_{T \to \infty} d_T(X).$$

In a similar way we may define *the modulus of increase* of function x and the set X (cf. [1]).

Finally, let us define the set quantity μ_d in the following way

$$\mu_d(X) \,= \omega_0^\infty(X) + d_\infty(X) + \limsup_{t \to \infty} \operatorname{diam} X(t). \tag{3.1}$$

It can be shown (see [7]) that μ_d is the measure of noncompactness in the algebra $BC(\mathbb{R}_+)$. The kernel $\ker \mu_d$ of this measure consists of all sets $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle $X(t)$ formed by functions from X tends to zero at infinity. Moreover, all functions from X are *ultimately nondecreasing* on $\mathbb{R}_+$ (cf. [1] for details).

Now we recall that the measure μ_d has following property.

Theorem 3.1. *The measure of noncompactness μ_d defined by* (3.1) *satisfies condition* (m) *on the family of all nonempty and bounded subsets X of Banach algebra $BC(\mathbb{R}_+)$ such that functions belonging to X are nonnegative on $\mathbb{R}_+$.*

The proof of above theorem may be found in [4].

Further, let us assume that Ω is a nonempty subset of the Banach algebra $BC(\mathbb{R}_+)$ and $F\colon \Omega \to BC(\mathbb{R}_+)$ is an operator. Consider the operator equation

$$x(t) = (Fx)(t), \quad t \in \mathbb{R}_+, \tag{3.2}$$

where $x \in \Omega$.

Definition 3.2. [1] We say that solutions of Eq. (3.2) are *asymptotically stable* if there exists a ball $B(x_0, r)$ in $BC(\mathbb{R}_+)$ such that $B(x_0, r) \cap \Omega \neq \phi$ and for each $\varepsilon > 0$ there exists $T > 0$ such that $|x(t) - y(t)| \leq \varepsilon$ for all solutions $x, y \in B(x_0, r) \cap \Omega$ of Eq. (3.2) and for $t \geq T$.

Let us mention that the measure of noncompactness μ_d defined by formula (3.1) allows us to characterize solutions of considered operator equations in terms of the concept of asymptotic stability. Namely, if solutions of an operator equation considered in the algebra $BC(\mathbb{R}_+)$ belong to a bounded subset being a member of the family $\ker \mu_d$, then from the previously given description of the kernel $\ker \mu_d$ we infer that those solutions are asymptotically stable in the sense of Definition 3.2 (cf. also Remark 2.5).

4. The existence of asymtotically stable and ultimately nondecreasing solutions of a quadratic fractional integral equation in the Banach algebra $BC(\mathbb{R}_+)$

In this section we will consider the solvability of some functional integral equation in the Banach algebra $BC(\mathbb{R}_+)$. Using the technique of measures of noncompactness we will prove that this equation has solutions on an unbounded interval. Moreover, we will also obtain some characterization of those solutions. Obviously, we will apply the measure of noncompactness described in the previous section.

In our considerations we will often use the so-called *superposition operator*. In order to define that operator we assume that $J \subset \mathbb{R}$ is an interval and consider a set X_J consisting of all functions $x\colon \mathbb{R}_+ \to J$. Moreover, $f\colon \mathbb{R}_+ \times J \to \mathbb{R}$ is a given function. Then, for every function $x \in X_J$ we may assign the function Fx defined by the formula

$$(Fx)(t) = f(t, x(t))$$

for $t \in \mathbb{R}_+$. The operator F defined in such a way is called the superposition operator generated by the function f (cf. [2]).

The below quoted lemma [1] presents a useful property of the superposition operator which is considered in the Banach space $B(\mathbb{R}_+)$ consisting of all real functions defined and bounded on $\mathbb{R}_+$. Obviously, the space $B(\mathbb{R}_+)$ is equipped with the classical supremum norm. Since $B(\mathbb{R}_+)$ has the structure of a Banach algebra, we can consider the Banach algebra $BC(\mathbb{R}_+)$ as a subalgebra of $B(\mathbb{R}_+)$.

Lemma 4.1. *Assume that the following hypotheses are satisfied:*

(α) *The function f is continuous on the set $\mathbb{R}_+ \times J$.*

(β) *The function $t \mapsto f(t,u)$ is ultimately nondecreasing uniformly with respect to u belonging to bounded subintervals of J, i.e.,*

$$\lim_{T\to\infty} \big\{ \sup\{|f(t,u) - f(s,u)| - [f(t,u) - f(s,u)] : t > s \geq T, u \in J_1\}\big\} = 0$$

for any bounded subinterval $J_1 \subseteq J$.

(γ) *For any fixed $t \in \mathbb{R}_+$ the function $u \mapsto f(t,u)$ is nondecreasing on J.*

(δ) *The function $u \mapsto f(t,u)$ satisfies a Lipschitz condition, i.e., there exist a constant $k > 0$ such that*

$$|f(t,u) - f(t,v)| \leq k|u - v|$$

for all $t \geq 0$ and all $u, v \in J$.

Then the inequality

$$d_\infty(Fx) \leq k d_\infty(x)$$

holds for any function $x \in X_J \cap B(\mathbb{R}_+)$, where k is the Lipschitz constant from assumption (δ).

Observe that in view of the remark mentioned previously the above lemma is also valid in the Banach algebra $BC(\mathbb{R}_+)$.

As we announced before, in this section we will investigate the existence of solutions of the quadratic fractional integral equation having the form

$$x(t) = (U_1x)(t)(U_2x)(t), \tag{4.1}$$

where

$$(U_ix)(t) = m_i(t) + f_i(t, x(t)) \int_0^t \frac{v_i(t,s,x(s))}{(t-s)^{\alpha_i}} ds \tag{4.2}$$

for $t \in \mathbb{R}_+$ and $i = 1, 2$. Here we assume that $\alpha_i \in (0,1)$ is a fixed number for $i = 1, 2$. Our investigations will be conducted in the Banach algebra $BC(\mathbb{R}_+)$. For further purposes we define the few operators on the space $BC(\mathbb{R}_+)$ by putting:

$$(F_ix)(t) = f_i(t, x(t)),$$

$$(V_i x)(t) = \int_0^t \frac{v_i(t, s, x(s))}{(t-s)^{\alpha_i}} ds$$

for $i = 1, 2$. Obviously, we have

$$(U_i x)(t) = m_i(t) + (F_i x)(t)(V_i x)(t)$$

for $i = 1, 2$ and for $t \in \mathbb{R}_+$. Moreover, let us introduce the following sets:

$$\Delta := \{(t, s) \in \mathbb{R}_+ \times \mathbb{R}_+ : s \leq t\},$$

$$A := \{(t, s, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} : s \leq t\},$$

$$A_+ := \{(t, s, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ : s \leq t\}.$$

We will study Eq. (4.1) under following assumptions.

(i) The function m_i is nonnegative, bounded, continuous and ultimately nondecreasing ($i = 1, 2$).

(ii) The function $f_i \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions $(\alpha) - (\gamma)$ of Lemma 4.1. Moreover, the function $t \mapsto f_i(t, 0)$ is bounded for $i = 1, 2$.

(iii) The function f_i ($i = 1, 2$) satisfies the Lipschitz condition with respect to the second variable, i.e., there exists a constant $k_i > 0$ such that

$$|f_i(t, x) - f_i(t, y)| \leq k_i |x - y|$$

for $x, y, t \in \mathbb{R}_+$ ($i = 1, 2$).

(iv) $v_i \colon A \to \mathbb{R}$ is a continuous function such that $v_i \colon A_+ \to \mathbb{R}_+$ ($i = 1, 2$). Moreover, there exists a continuous and nondecreasing function $G_i \colon \mathbb{R}_+ \to \mathbb{R}_+$ and a bounded and continuous function $g_i \colon \Delta \to \mathbb{R}_+$ such that $v_i(t, s, x) = g_i(t, s) G_i(|x|)$ for $t, s \in \mathbb{R}_+$ ($s \leq t$), $x \in \mathbb{R}$ and $i = 1, 2$.

(v) The following property holds

$$\lim_{t \to \infty} \int_0^t \frac{g_i(t, s)}{(t-s)^{\alpha_i}} ds = 0$$

for $i = 1, 2$.

In view of above assumptions we may define following constants ($i = 1, 2$):

$$\overline{F}_i = \sup\{|f_i(t, 0)| : t \in \mathbb{R}_+\},$$

$$\overline{G}_i = \sup\left\{ \int_0^t \frac{g_i(t, s)}{(t-s)^{\alpha_i}} ds : t \in \mathbb{R}_+ \right\},$$

$$\overline{g}_i = \sup\{g_i(t, s) : t, s \in \mathbb{R}_+\},$$

$$\overline{F} = \max\{\overline{F}_1, \overline{F}_2\},$$

$$k = \max\{k_1, k_2\},$$

$$m = \max\{\|m_1\|, \|m_2\|\}.$$

The last assumption has the form:

(vi) There exists a solution $r_0 > 0$ of the inequality

$$[m + k\overline{G}_1 r G_1(r) + \overline{F}\,\overline{G}_1 G_1(r)][m + k\overline{G}_2 r G_2(r) + \overline{F}\,\overline{G}_2 G_2(r)] \leq r$$

such that

$$mk(\overline{G}_1 G_1(r_0) + \overline{G}_2 G_2(r_0)) + 2k\overline{F}\,\overline{G}_1 G_1(r_0)\overline{G}_2 G_2(r_0)$$
$$+ 2k^2 r_0 \overline{G}_1 G_1(r_0)\overline{G}_2 G_2(r_0) < 1.$$

The existence result concerning the functional integral equation (4.1) is contained in the below given theorem.

Theorem 4.2. *Under assumptions (i)-(vi) Eq. (4.1) has at least one solution $x = x(t)$ in the space $BC(\mathbb{R}_+)$ which is nonnegative, asymptotically stable and ultimately nondecreasing.*

Proof. Let us assume that Ω is the subset of the Banach algebra $BC(\mathbb{R}_+)$ consisting of all functions being nonnegative on $\mathbb{R}_+$. We will consider operators V_i $(i = 1, 2)$ on the set Ω. Now, fix an arbitrary function $x \in \Omega$. Then, in virtue of assumptions (i), (ii) and (iv) we derive that the function $U_i x$ is nonnegative on $\mathbb{R}_+$ $(i = 1, 2)$.

Next, in view of (4.2) and the imposed assumptions, we obtain:

$$\begin{aligned}(U_i x)(t) &\leq m_i(t) + [k_i x(t) + f_i(t, 0)] \int_0^t \frac{v_i(t, s, x(s))}{(t-s)^{\alpha_i}} ds \\ &\leq m_i(t) + [k_i x(t) + f_i(t, 0)] G_i(\|x\|) \int_0^t \frac{g_i(t, s)}{(t-s)^{\alpha_i}} ds \\ &\leq \|m_i\| + k_i \overline{G}_i \|x\| G_i(\|x\|) + \overline{F}\,\overline{G}_i G_i(\|x\|) \qquad (4.3)\end{aligned}$$

for $t \in \mathbb{R}_+$, $i = 1, 2$.
This estimate yields that the function $U_i x$ is bounded on $\mathbb{R}_+$ $(i = 1, 2)$.

Next, let us observe that in view of the properties of the superposition operator [2] and assumption (ii) we infer that the function $F_i x$ is continuous on $\mathbb{R}_+$ $(i = 1, 2)$. Therefore, in order to show that $U_i x$ is continuous on the interval $\mathbb{R}_+$ it is sufficient to show that the function $V_i x$ is continuous on $\mathbb{R}_+$.

To this end let us fix $T > 0$ and $\varepsilon > 0$ and choose arbitrarily $t, s \in [0, T]$ such that

$|t-s| \le \varepsilon$. Without loss of generality we may assume that $s < t$. Then we obtain

$$\begin{aligned}
|(V_i x)(t) - (V_i x)(s)| &\le \Big| \int_0^t \frac{v_i(t,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau - \int_0^t \frac{v_i(s,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau \Big| \\
&+ \Big| \int_0^t \frac{v_i(s,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau - \int_0^s \frac{v_i(s,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau \Big| \\
&+ \Big| \int_0^s \frac{v_i(s,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau - \int_0^s \frac{v_i(s,\tau,x(\tau))}{(s-\tau)^{\alpha_i}} d\tau \Big| \\
&\le \int_0^t \frac{|v_i(t,\tau,x(\tau)) - v_i(s,\tau,x(\tau))|}{(t-\tau)^{\alpha_i}} d\tau + \int_s^t \frac{v_i(s,\tau,x(\tau))}{(t-\tau)^{\alpha_i}} d\tau \\
&+ \int_0^s v_i(s,\tau,x(\tau)) \left[\frac{1}{(s-\tau)^{\alpha_i}} - \frac{1}{(t-\tau)^{\alpha_i}} \right] d\tau \\
&\le \omega^T_{\|x\|}(v_i,\varepsilon) \int_0^t \frac{1}{(t-\tau)^{\alpha_i}} d\tau + G_i(\|x\|)\overline{g}_i \int_s^t \frac{1}{(t-\tau)^{\alpha_i}} d\tau \\
&+ G_i(\|x\|)\overline{g}_i \int_0^s \left[\frac{1}{(s-\tau)^{\alpha_i}} - \frac{1}{(t-\tau)^{\alpha_i}} \right] d\tau \\
&\le \omega^T_{\|x\|}(v_i,\varepsilon) \frac{t^{1-\alpha_i}}{1-\alpha_i} + G_i(\|x\|)\overline{g}_i \frac{(t-s)^{1-\alpha_i}}{1-\alpha_i} \\
&+ G_i(\|x\|)\overline{g}_i \left[\frac{s^{1-\alpha_i}}{1-\alpha_i} - \frac{t^{1-\alpha_i}}{1-\alpha_i} + \frac{(t-s)^{1-\alpha_i}}{1-\alpha_i} \right] \\
&\le \omega^T_{\|x\|}(v_i,\varepsilon) \frac{T^{1-\alpha_i}}{1-\alpha_i} + 2G_i(\|x\|)\overline{g}_i \frac{\varepsilon^{1-\alpha_i}}{1-\alpha_i}, \qquad (4.4)
\end{aligned}$$

where we denoted
$\omega^T_d(v_i,\varepsilon) = \sup\{|v_i(t,\tau,x) - v_i(s,\tau,x)| : t,s,\tau \in [0,T],\ |t-s| \le \varepsilon,\ x \in [-d,d]\}$.

From the above estimate and the fact that function v_i is uniformly continuous on the set $\{(t,s,y) \in \mathbb{R}^3 : 0 \le s \le t \le T,\ y \in [-\|x\|, \|x\|]\}$ we have that the last part of above estimate tends to zero as $\varepsilon \to 0$ and this implies the continuity of the function $V_i x$. Gathering the above established facts and estimate (4.4) we conclude that the operator U_i $(i = 1,2)$ transforms the set Ω into itself.

Apart from this, in view of (4.3) and assumption (vi) we infer that there exists a number $r_0 > 0$ such that the operator $S = U_1 U_2$ transforms into itself the set Ω_{r_0} defined in the following way

$$\Omega_{r_0} = \{x \in BC(\mathbb{R}_+) : 0 \le x(t) \le r_0 \text{ for } t \in \mathbb{R}_+\}.$$

Moreover, the following inequality is satisfied

$$\|U_i \Omega_{r_0}\| \le m + k_i \overline{G} r_0 G_i(r_0) + \overline{F}\,\overline{G} G_i(r_0).$$

In the sequel we will work with the measure of noncompactness μ_d. Thus, let us fix a nonempty subset X of the set Ω_{r_0} and choose arbitrary numbers $T > 0$ and

$\varepsilon > 0$. Then, for $x \in X$ and for $t, s \in [0,T]$ such that $|t-s| \le \varepsilon$ and $t \ge s$ we have

$$\begin{aligned} |(U_i x)(t) - (U_i x)(s)| &\le \omega^T(m_i,\varepsilon) + |(F_i x)(t)(V_i x)(t) - (F_i x)(s)(V_i x)(s)| \\ &\le \omega^T(m_i,\varepsilon) + |(F_i x)(t)(V_i x)(t) - (F_i x)(s)(V_i x)(t)| \\ &+ |(F_i x)(s)(V_i x)(t) - (F_i x)(s)(V_i x)(s)| \\ &\le \omega^T(m_i,\varepsilon) + |(F_i x)(t) - (F_i x)(s)||(V_i x)(t)| \\ &+ |(F_i x)(s)||(V_i x)(t) - (V_i x)(s)|. \end{aligned} \tag{4.5}$$

Reasoning similarly we derive the estimate

$$\begin{aligned} |(F_i x)(t) - (F_i x)(s)| &\le |f_i(t,x(t)) - f_i(t,x(s))| + |f_i(t,x(s)) - f_i(s,x(s))| \\ &\le k_i|x(t)-x(s)| + |f_i(t,x(s)) - f_i(s,x(s))| \\ &\le k_i\omega^T(x,\varepsilon) + \omega^T_{\|x\|}(f_i,\varepsilon), \end{aligned} \tag{4.6}$$

where we denoted
$\omega^T_d(f_i,\varepsilon) = \sup\{|f_i(t,x) - f_i(s,x)| : \ t,s \in [0,T],\ |t-s| \le \varepsilon,\ x \in [-d,d]\}$.
Moreover, we have the following evaluations

$$|(V_i x)(t)| \le G_i(\|x\|)\int_0^t \frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}d\tau \le G_i(\|x\|)\overline{G}_i,$$

$$|(F_i x)(s)| \le k_i|x(s)| + |f_i(s,0)| \le k_i r_0 + \overline{F}_i,$$

which hold for arbitrary $t, s \in \mathbb{R}_+$ and $i = 1, 2$.

Further, linking (4.5), (4.6) with the above obtained evaluations, we infer that the following estimate holds

$$\begin{aligned} |(U_i x)(t) - (U_i x)(s)| &\le \omega^T(m_i,\varepsilon) + \big[k_i\omega^T(x,\varepsilon) + \omega^T_{\|x\|}(f_i,\varepsilon)\big]G_i(\|x\|)\overline{G}_i \\ &+ \big[k_i r_0 + \overline{F}_i\big]\left[\omega^T_{\|x\|}(v_i,\varepsilon)\frac{T^{1-\alpha_i}}{1-\alpha_i} + 2G_i(\|x\|)\overline{g}_i\frac{\varepsilon^{1-\alpha_i}}{1-\alpha_i}\right]. \end{aligned}$$

Observe that the terms $\omega^T(m_i,\varepsilon)$, $\omega^T_{\|x\|}(f_i,\varepsilon)$ and $\omega^T_{\|x\|}(v_i,\varepsilon)$ tend to zero as $\varepsilon \to 0$ since the functions m_i, f_i and v_i are uniformly continuous on the sets $[0,T]$, $[0,T] \times [-\|x\|,\|x\|]$ and $\{(t,s,y) \in \mathbb{R}^3 : 0 \le s \le t \le T,\ y \in [-\|x\|,\|x\|]\}$, respectively. Hence we obtain

$$\omega^T_0(U_i X) \le k_i\overline{G}_i G_i(r_0)\omega^T_0(X)$$

and consequently

$$\omega^\infty_0(U_i X) \le k_i\overline{G}_i G_i(r_0)\omega^\infty_0(X). \tag{4.7}$$

In what follows, let us choose arbitrarily $x, y \in X$ and $t \in \mathbb{R}_+$. Then, applying our assumptions, we derive

$$
\begin{aligned}
&|(U_ix)(t)-(U_iy)(t)| \\
&\leq |f_i(t,x(t))-f_i(t,y(t))|\int_0^t \frac{v_i(t,\tau,x(\tau))}{(t-\tau)^{\alpha_i}}d\tau \\
&+f_i(t,y(t))\int_0^t \frac{|v_i(t,\tau,x(\tau))-v_i(t,\tau,y(\tau))|}{(t-\tau)^{\alpha_i}}d\tau \\
&\leq k_i|x(t)-y(t)|G_i(\|x\|)\int_0^t \frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}d\tau \\
&+[k_iy(t)+f_i(t,0)]\int_0^t \frac{g_i(t,\tau)[G_i(x(\tau))-G_i(y(\tau))]}{(t-\tau)^{\alpha_i}}d\tau \\
&\leq k_i|x(t)-y(t)|G_i(r_0)\int_0^t \frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}d\tau+[k_ir_0+\overline{F}_i]2G_i(r_0)\int_0^t \frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}d\tau.
\end{aligned}
$$

Hence, keeping in mind of assumption (v), we obtain the following equality

$$
\limsup_{t\to\infty} \operatorname{diam}(U_iX)(t)=0. \tag{4.8}
$$

Now, we show that U_i is continuous on the set Ω_{r_0}. To this end fix $\varepsilon>0$ and take $x,y\in\Omega_{r_0}$ such that $\|x-y\|\leq\delta$. In view of (4.8) we know that we may find a number $T>0$ such that for arbitrary $t\geq T$ we get $|(U_ix)(t)-(U_iy)(t)|\leq\varepsilon$. On the other hand, if we take $t\in[0,T]$, we have

$$
\begin{aligned}
|(U_ix)(t)-(U_iy)(t)| &\leq |f_i(t,x(t))-f_i(t,y(t))|\int_0^t \frac{v_i(t,\tau,x(\tau))}{(t-\tau)^{\alpha_i}}d\tau \\
&+ f_i(t,x(t))\int_0^t \frac{|v_i(t,\tau,x(\tau))-v_i(t,\tau,y(\tau))|}{(t-\tau)^{\alpha_i}}d\tau \\
&\leq k_i|x(t)-y(t)|G_i(\|x\|)\int_0^t \frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}d\tau \\
&+ [k_i|y(t)|+f_i(t,0)]\xi^T_{r_0}(v_i,\delta)\int_0^t \frac{d\tau}{(t-\tau)^{\alpha_i}} \\
&\leq k\delta\overline{G}_iG_i(r_0)+(kr_0+\overline{F})\frac{T^{1-\alpha_i}}{1-\alpha_i}\xi^T_{r_0}(v_i,\delta),
\end{aligned}
$$

where we denoted
$\xi^T_d(v_i,\delta)=\sup\{|v_i(t,s,x)-v_i(t,s,y)|:\ t,s\in[0,T],x,y\in[-d,d],|x-y|\leq\delta\}$.

In view of the uniform continuity of the function v_i on the set $\{(t,s,y)\in\mathbb{R}^3:0\leq s\leq t\leq T,\ y\in[-r_0,r_0]\}$ we have that $\xi^T_{r_0}(v_i,\delta)\to 0$ as $\delta\to 0$. This yields that the last term in the above estimate is sufficiently small for $i=1,2$.

Next, fix arbitrarily $T>0$ and choose t,s such that $t>s\geq T$. Then we have

$$
0\leq\left|\frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}-\frac{g_i(s,\tau)}{(s-\tau)^{\alpha_i}}\right|-\left[\frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}-\frac{g_i(s,\tau)}{(s-\tau)^{\alpha_i}}\right]\leq 2\frac{g_i(s,\tau)}{(s-\tau)^{\alpha_i}}.
$$

Using (v) we obtain

$$\lim_{T\to\infty}\left\{\sup\left\{\int_0^s\left\{\left|\frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}-\frac{g_i(s,\tau)}{(s-\tau)^{\alpha_i}}\right|-\left[\frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}-\frac{g_i(s,\tau)}{(s-\tau)^{\alpha_i}}\right]\right\}d\tau : T\le s<t\right\}\right\}=0. \tag{4.9}$$

Then, for an arbitrary $x\in X$ we obtain

$$\begin{aligned}
&|(U_ix)(t)-(U_ix)(s)|-[(U_ix)(t)-(U_ix)(s)]\\
&\le |m_i(t)-m_i(s)|-[m_i(t)-m_i(s)]+|(F_ix)(t)(V_ix)(t)-(F_ix)(s)(V_ix)(t)|\\
&+|(F_ix)(s)(V_ix)(t)-(F_ix)(s)(V_ix)(s)|-[(F_ix)(t)(V_ix)(t)-(F_ix)(s)(V_ix)(t)]\\
&-[(F_ix)(s)(V_ix)(t)-(F_ix)(s)(V_ix)(s)]\\
&\le d_T(m_i)+d_T(F_ix)(V_ix)(t)\\
&+(F_ix)(s)\big\{|(V_ix)(t)-(V_ix)(s)|-[(V_ix)(t)-(V_ix)(s)]\big\}.
\end{aligned} \tag{4.10}$$

On the other hand we get

$$\begin{aligned}
&|(V_ix)(t)-(V_ix)(s)|-[(V_ix)(t)-(V_ix)(s)]\\
&\le\left|\int_0^s\frac{g_i(t,\tau)G_i(x(\tau))}{(t-\tau)^{\alpha_i}}d\tau-\int_0^s\frac{g_i(s,\tau)G_i(x(\tau))}{(s-\tau)^{\alpha_i}}d\tau\right|+\left|\int_s^t\frac{g_i(t,\tau)G_i(x(\tau))}{(t-\tau)^{\alpha_i}}d\tau\right|\\
&-\left[\int_0^s\frac{g_i(t,\tau)G_i(x(\tau))}{(t-\tau)^{\alpha_i}}d\tau-\int_0^s\frac{g_i(s,\tau)G_i(x(\tau))}{(s-\tau)^{\alpha_i}}d\tau\right]-\int_s^t\frac{g_i(t,\tau)G_i(x(\tau))}{(t-\tau)^{\alpha_i}}d\tau\\
&\le\int_0^s\left|\frac{g_i(t,\tau)G_i(x(\tau))}{(t-\tau)^{\alpha_i}}d\tau-\frac{g_i(s,\tau)G_i(x(\tau))}{(s-\tau)^{\alpha_i}}\right|d\tau\\
&-\int_0^s\left[\frac{g_i(t,\tau)G_i(x(\tau))}{(t-\tau)^{\alpha_i}}d\tau-\frac{g_i(s,\tau)G_i(x(\tau))}{(s-\tau)^{\alpha_i}}\right]d\tau\\
&\le G_i(\|x\|)\int_0^s\left\{\left|\frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}-\frac{g_i(s,\tau)}{(s-\tau)^{\alpha_i}}\right|-\left[\frac{g_i(t,\tau)}{(t-\tau)^{\alpha_i}}-\frac{g_i(s,\tau)}{(s-\tau)^{\alpha_i}}\right]\right\}d\tau.
\end{aligned}$$

Now, taking into account assumptions (i), (iv) and estimates (4.9), (4.10), we derive

$$d_\infty(U_ix)\le d_\infty(F_ix)G_i(r_0)\overline{G}_i$$

for $i=1,2$. Hence, in view of Lemma 4.1, we derive the following inequality

$$d_\infty(U_ix)\le k_i\overline{G}_iG_i(r_0)d_\infty(x)$$

for $i=1,2$. Further, combining the above inequality an (4.6), (4.7), (4.10), we obtain

$$\mu_d(U_iX)\le k_i\overline{G}_iG_i(r_0)\mu_d(X)$$

for $i=1,2$.

Therefore, applying Theorem 2.4 we derive that the operator $S=U_1U_2$ is a contraction with respect to the measure of noncompactness μ_d with the constant L given by the formula

$$L=mk(\overline{G}_1G_1(r_0)+\overline{G}_2G_2(r_0))+2k\overline{F}\,\overline{G}_1G_1(r_0)\overline{G}_2G_2(r_0)+2k^2r_0\overline{G}_1G_1(r_0)\overline{G}_2G_2(r_0).$$

Observe that assumption (vi) implies that $L < 1$. Thus, taking into account Theorem 2.4 we infer that the operator S has at least one fixed point $x = x(t)$ belonging to the set Ω_{r_0}. Moreover, in view of Remark 2.5 we conclude that x is nonnegative on $\mathbb{R}_+$, asymptotically stable and ultimately nondecreasing.
This completes the proof. □

Now we provide an example illustrating Theorem 4.2.

Example 4.3. Consider the quadratic fractional integral equation having form (4.1) with the operators U_1, U_2 defined by the following formulas

$$(U_1x)(t) = \frac{t}{3t+1} + \arctan(t^2 + x(t)) \int_0^t \frac{e^{-(t+s)}\sqrt{|x(t)|}}{(t-s)^{1/3}} ds,$$

$$(U_2x)(t) = \frac{1-e^{-t}}{4} + \frac{1}{2}\ln(x(t)+1) \int_0^t \frac{8x^4(t)}{5(t+s+2)^3(t-s)^{1/5}} ds$$

for $t \in \mathbb{R}_+$.

Observe that in this case the functions involved in Eq. (4.1) have the form:

$$m_1(t) = \frac{t}{3t+1}, \qquad m_2(t) = \frac{1-e^{-t}}{4},$$

$$f_1(t,x) = \arctan(t^2 + x), \qquad f_2(t,x) = \frac{1}{2}\ln(x+1),$$

$$v_1(t,s,x) = e^{-(t+s)}\sqrt{|x|},$$

$$v_2(t,s,x) = \frac{8x^4}{5(t+s+2)^3}.$$

Moreover, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{5}$.

It is easy to check that for the above functions there are satisfied assumptions of Theorem 4.2. Indeed, we have that the function $m_i = m_i(t)$ is nonnegative, bounded and continuous on $\mathbb{R}_+$ $(i = 1, 2)$. Since m_1 and m_2 are increasing on $\mathbb{R}_+$ we derive that they are also ultimately nondecreasing on $\mathbb{R}_+$. Moreover, $\|m_1\| = \frac{1}{3}$ and $\|m_2\| = \frac{1}{4}$. Thus, there is satisfied assumption (i). Further notice that the functions f_i $(i = 1, 2)$ transform continuously the set $\mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$. Moreover, f_1 is nondecreasing with respect to both variables and satisfies the Lipschitz condition (with respect to the second variable) with the constant $k_1 = 1$. Similarly, the function $f_2 = f_2(t, x)$ is increasing with respect to x and satisfies the Lipschitz condition with the constant $k_2 = \frac{1}{2}$. Apart from this it is easily seen that $\overline{F}_1 = \frac{\pi}{2}$, $\overline{F}_2 = 0$. Summing up, we see that functions f_1 and f_2 satisfy assumptions (ii) and (iii).

Next, let us note that the function v_i is continuous on the set A and transforms the set A_+ into $\mathbb{R}_+$ for $i = 1, 2$. Apart from this the function v_i can be represented in the form $v_i(t,s,x) = g_i(t,s)G_i(|x|)$ $(i = 1, 2)$, where $g_1(t,s) = e^{-(t+s)}$, $G_1(x) = \sqrt{|x|}$, $g_2(t,s) = \frac{8}{5(t+s+2)^3}$, $G_2(x) = x^4$. It is easily seen that assumption (iv) is satisfied for the functions v_1 and v_2.

Further on, we have

$$\int_0^t \frac{g_1(t,s)}{(t-s)^{\alpha_1}}ds = \int_0^t \frac{e^{-t-s}}{(t-s)^{1/3}}ds \le e^{-t}\int_0^t \frac{ds}{(t-s)^{1/3}} = \frac{3}{2}\,e^{-t}\,t^{2/3}.$$

Hence we see that

$$\lim_{t\to\infty}\int_0^t \frac{g_1(t,s)}{(t-s)^{\alpha_1}}ds = 0.$$

Moreover, we get

$$\int_0^t \frac{g_2(t,s)}{(t-s)^{\alpha_2}}ds = \int_0^t \frac{8}{5(t+s+2)^3(t-s)^{1/5}}ds \le \frac{8}{5(t+2)^3}\int_0^t \frac{ds}{(t-s)^{1/5}} = \frac{2t^{4/5}}{(t+2)^3}.$$

Thus, we have

$$\lim_{t\to\infty}\int_0^t \frac{g_2(t,s)}{(t-s)^{\alpha_2}}ds = 0.$$

This shows that assumption (v) is satisfied.

Finally, let us notice that taking into account the above estabilished facts we have that $m = \max\{\|m_1\|, \|m_2\|\} = \frac{1}{3}$, $k = \max\{k_1, k_2\} = 1$ and $\overline{F} = \max\{\overline{F}_1, \overline{F}_2\} = \frac{\pi}{2}$. Thus, the first inequality from assumption (vi) has the form

$$[\frac{1}{3} + \overline{G}_1(r\sqrt{r} + \frac{\pi}{2}r)][\frac{1}{3} + \overline{G}_2(r^5 + \frac{\pi}{2}r^4)] \le r.$$

It can be shown that the number $r_0 = \frac{1}{2}$ is a solution of the above inequality such that it satisfies also the second inequality from assumption (vi).

Applying Theorem 4.2 we infer that the quadratic fractional integral equation considered in this example has a solution belonging to the set

$$\Omega_{\frac{1}{2}} = \{x \in BC(\mathbb{R}_+) :\, 0 \le x(t) \le \frac{1}{2} \text{ for } t \in \mathbb{R}_+\},$$

which is asymtotically stable and ultimately nondecreasing.

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Weak Solutions of Fractional Order Differential Equations via Volterra-Stieltjes Integral Operator

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Abstract: The fractional derivative of the Riemann-Liouville and Caputo types played an important role in the development of the theory of fractional derivatives, integrals and for its applications in pure mathematics ([18], [21]). In this paper, we study the existence of weak solutions for fractional differential equations of Riemann-Liouville and Caputo types. We depend on converting of the mentioned equations to the form of functional integral equations of Volterra-Stieltjes type in reflexive Banach spaces.

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1. Introduction and preliminaries

Let E be a reflexive Banach space with norm $\| \, . \, \|$ and dual E^*. Denote by $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.
Fractional differential equations have received increasing attention due to its applications in physics, chemistry, materials, engineering, biology, finance [15, 16]. Fractional order derivatives have the memory property and can describe many phenomena that integer order derivatives cant characterize. Only a few papers consider fractional differential equations in reflexive Banach spaces with the weak topology [6, 7, 14, 22, 23].
Here we study the existence of weak solutions of the Volterra-Stieltjes integral equation

$$x(t) \; = \; p(t) + \int_0^t f(s, x(s)) \; d_s g(t, s), \quad t \in I = [0, T],$$

in the reflexive Banach space E.
Let $\alpha \in (0,1)$. As applications, we study the existence of weak solution for the differential equations of fractional order

$$ {}^{R}D^{\alpha}x(t) = f(t, x(t)), \ t \in (0,T] \tag{1.1} $$

with the initial data

$$ x(0) = 0 \ , \tag{1.2} $$

where ${}^{R}D^{\alpha}x(.)$ is a Riemann-Liouville fractional derivative of the function $x : I = [0,T] \to E$.
Also we study the existence of mild solution for the initial value problem

$$ {}^{C}D^{\alpha}x(t) = f(t, x(t)), \ t \in (0,T] \tag{1.3} $$

with the initial data

$$ x(0) = x_0 \ , \tag{1.4} $$

where ${}^{C}D^{\alpha}x(.)$ is a Caputo fractional derivative of the function $x : I : [0,T] \to E$.
Functional integral equations of Volterra-Stieltjes type have been studied in the space of continuous functions in many papers for example, (see [1-5] and [8]).
For the properties of the Stieltjes integral (see Banaś [1]).

Definition 1.1. The fractional (arbitrary) order integral of the function $f \in L_1$ of order $\alpha > 0$ is defined as [18, 21]

$$ I^{\alpha}f(t) := \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \ ds. $$

For the fractional-order derivative we have the following two definitions.

Definition 1.2. The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0,1)$ is defined as ([18], [21])

$$ {}^{R}D_a^{\alpha}f(t) = \frac{d}{dt}\int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s) \ ds $$

or

$$ {}^{R}D_a^{\alpha}f(t) = \frac{d}{dt} I_a^{1-\alpha} f(t). $$

Definition 1.3. The Caputo fractional-order derivative of $g(t)$ of order $\alpha \in (0,1]$ of the absolutely continuous function $g(t)$ is defined as ([9])

$$ {}^{C}D_a^{\alpha}g(t) = \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \frac{d}{ds} g(s) \ ds $$

or

$$ {}^{C}D_a^{\alpha}g(t) = I_a^{1-\alpha} \frac{d}{dt} g(t). $$

Now, we shall present some auxiliary results that will be need in this work. Let E be a Banach space (need not be reflexive) and let $x : [a, b] \to E$, then

(1-) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in [a, b]$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at t_0.

(2-) A function $h : E \to E$ is said to be weakly sequentially continuous if h maps weakly convergent sequences in E to weakly convergent sequences in E.

If x is weakly continuous on I, then x is strongly measurable and hence weakly measurable (see [10] and [13]). It is evident that in reflexive Banach spaces, if x is weakly continuous function on $[a, b]$, then x is weakly Riemann integrable (see [13]).

Definition 1.4. Let $f : I \times E \to E$. Then $f(t, u)$ is said to be weakly-weakly continuous at (t_0, u_0) if given $\epsilon > 0$, $\phi \in E^*$ there exists $\delta > 0$ and a weakly open set U containing u_0 such that

$$| \phi(f(t, u) - f(t_0, u_0)) | < \epsilon$$

whenever

$$| t - t_0 | < \delta \text{ and } u \in U.$$

Now, we have the following fixed point theorem, due to O'Regan, in the reflexive Banach space (see [19]) and some propositions which will be used in the sequel [13, 20].

Theorem 1.5. *Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of $C[I, E]$ and let $F : Q \to Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in E for each $t \in I$. Then, F has a fixed point in the set Q.*

Proposition 1.6. *A convex subset of a normed space E is closed if and only if it is weakly closed.*

Proposition 1.7. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Proposition 1.8. *Let E be a normed space with $y \in E$ and $y \neq 0$. Then there exists a $\phi \in E^*$ with $\| \phi \| = 1$ and $\| y \| = \phi(y)$.*

2. Volterra-Stieltjes integral equation

In this section we prove the existence of weak solutions for the Volterra-Stieltjes integral equation

$$x(t) = p(t) + \int_0^t f(s, x(s))\, d_s g(t, s), \quad t \in I = [0, T], \tag{2.5}$$

in the space $C[I,E]$. To facilitate our discussion, denote Λ by

$$\Lambda = \{(t,s) : 0 \leq s \leq t \leq T\}$$

and let $p : I \to E,\ f : I \times E \to E$ and $g : \Lambda \to R$ be functions such that:

(i) $p \in C[I,E]$.

(ii) The function f is weakly-weakly continuous.

(iii) There exists a constant M such that $\| f(t,x) \| \leq M$.

(iv) The function g is continuous on Λ.

(v) The function $s \to g(t,s)$ is of bounded variation on $[0,t]$ for each fixed $t \in I$.

(vi) For any $\epsilon > 0$ there exists $\delta > 0$ for all $t_1, t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 \leq \delta$ the following inequality holds

$$\bigvee_{0}^{t_1} [g(t_2,s) - g(t_1,s)] \leq \epsilon.$$

(vii) $g(t,0) = 0$ for any $t \in I$.

Obviously we will assume that g satisfies assumptions (iv)-(vi). For our purposes we will only need the following lemmas.

Lemma 2.1. *[5] The function* $z \to \bigvee_{s=0}^{z} g(t,s)$ *is continuous on* $[0,t]$ *for any fixed* $t \in I$.

Lemma 2.2. *[5] For an arbitrary fixed* $0 < t_2 \in I$ *and for any* $\epsilon > 0$, *there exists* $\delta > 0$ *such that if* $t_1 \in I,\ t_1 < t_2$ *and* $t_2 - t_1 \leq \delta$ *then*

$$\bigvee_{s=t_1}^{t_2} g(t_2,s) \leq \epsilon.$$

Lemma 2.3. *[5] The function* $t \to \bigvee_{s=0}^{t} g(t,s)$ *is continuous on* I. *Then there exists a finite positive constant* K *such that*

$$K = \sup\{\bigvee_{s=0}^{t} g(t,s) : t \in I\}.$$

Definition 2.4. By a weak solution to (2.5) we mean a function $x \in C[I,E]$ which satisfies the integral equation (2.5). This is equivalent to find $x \in C[I,E]$ with

$$\phi(x(t)) \;=\; \phi(p(t) + \int_0^t f(s,x(s))\ d_s g(t,s)),\ t \in I\ \forall\ \phi \in E^*.$$

Now we can prove the following theorem.

Theorem 2.5. *Under the assumptions (i)-(vii), the Volterra-Stieltjes integral equation (2.5) has at least one weak solution* $x \in C[I, E]$.

Proof. Define the nonlinear Volterra-Stieltjes integral operator A by

$$Ax(t) = p(t) + \int_0^t f(s, x(s)) \, d_s g(t, s), \quad t \in I.$$

For every $x \in C[I, E]$, $f(., x(.))$ is weakly continuous ([24]). To see this we equip E and $I \times E$ with weak topology and note that $t \mapsto (t, x(t))$ is continuous as a mapping from I into $I \times E$, then $f(., x(.))$ is a composition of this mapping with f and thus for each weakly continuous $x : I \to E$, $f(., x(.)) : I \to E$ is weakly continuous, means that $\phi(f(., x(.)))$ is continuous, for every $\phi \in E^*$, g is of bounded variation. Hence $f(., x(.))$ is weakly Riemann-Stieltjes integrable on I with respect to $s \to g(t, s)$. Thus A makes sense.
For notational purposes $\| x \|_0 = \sup_{t \in I} \| x(t) \|$.
Now, define the set Q by

$$Q = \Big\{ x \in C[I, E] : \| x \|_0 \leq M_0 \ ,$$

$$\| x(t_2) - x(t_1) \| \leq \| p(t_2) - p(t_1) \| + MN(\epsilon) + M \bigvee_{s=t_1}^{t_2} g(t_2, s) \Big\} \ .$$

First notice that Q is convex and norm closed. Hence Q is weakly closed by Proposition 1.6.
Note that A is well defined, to see that, Let $t_1, \ t_2 \in I, \ t_2 > t_1$, without loss of generality, assume $Ax(t_2) - Ax(t_1) \neq 0$

$$\begin{aligned}
\| Ax(t_2) - Ax(t_1) \| &= \phi(Ax(t_2) - Ax(t_1)) \leq \ | \phi(p(t_2) - p(t_1)) | \\
&+ \ | \int_0^{t_2} \phi(f(s, x(s))) \, d_s g(t_2, s) - \int_0^{t_1} \phi(f(s, x(s))) \, d_s g(t_1, s) | \\
&\leq \ \| p(t_2) - p(t_1) \| + | \int_0^{t_1} \phi(f(s, x(s))) \, d_s g(t_2, s) \\
&+ \ \int_{t_1}^{t_2} \phi(f(s, x(s))) \, d_s g(t_2, s) - \int_0^{t_1} \phi(f(s, x(s))) \, d_s g(t_1, s) | \\
&\leq \ \| p(t_2) - p(t_1) \| + | \int_0^{t_1} \phi(f(s, x(s))) \, d_s [g(t_2, s) - g(t_1, s)] | \\
&+ \ | \int_{t_1}^{t_2} \phi(f(s, x(s))) \, d_s g(t_2, s) |
\end{aligned}$$

$$
\begin{aligned}
&\leq \ \| p(t_2) - p(t_1) \| \\
&+ \ \int_0^{t_1} | \phi(f(s, x(s))) | \ d_s[\bigvee_{z=0}^{s} (g(t_2, z) - g(t_1, z))] \\
&+ \ \int_{t_1}^{t_2} | \phi(f(s, x(s))) | \ d_s[\bigvee_{z=0}^{s} g(t_2, z)] \\
&\leq \ \| p(t_2) - p(t_1) \| + M \int_0^{t_1} d_s[\bigvee_{z=0}^{s} (g(t_2, z) - g(t_1, z))] \\
&+ \ M \int_{t_1}^{t_2} d_s[\bigvee_{z=0}^{s} g(t_2, z)] \\
&\leq \ \| p(t_2) - p(t_1) \| + M \bigvee_{s=0}^{t_1} (g(t_2, s) - g(t_1, s)) \\
&+ \ M[\bigvee_{s=0}^{t_2} g(t_2, s) - \bigvee_{s=0}^{t_1} g(t_2, s)] \\
&\leq \ \| p(t_2) - p(t_1) \| + MN(\epsilon) + M \bigvee_{s=t_1}^{t_2} g(t_2, s) \ ,
\end{aligned}
$$

where

$$
N(\epsilon) = \sup\{\bigvee_{s=0}^{t_1} (g(t_2, s) - g(t_1, s)) : t_1, t_2 \in I, \ t_1 < t_2, \ t_2 - t_1 \leq \epsilon\}.
$$

Hence

$$
\| Ax(t_2) - Ax(t_1) \| \leq \| p(t_2) - p(t_1) \| + MN(\epsilon) + M \bigvee_{s=t_1}^{t_2} g(t_2, s), \tag{2.6}
$$

and so $Ax \in C[I, E]$. We claim that $A : Q \to Q$ is weakly sequentially continuous and $A(Q)$ is weakly relatively compact. Once the claim is established, Theorem 1.5 guarantees the existence of a fixed point $x \in C[I, E]$ of the operator A and the integral equation (2.5) has a solution $x \in C[I, E]$.
To prove our claim, we start by showing that $A : Q \to Q$. Take $x \in Q$, note that the inequality (2.6) shows that AQ is norm continuous. Then by using Proposition 1.8

we get

$$\begin{aligned}
\| Ax(t) \| &= \phi(Ax(t)) \leq \ | \phi(p(t)) | + | \phi(\int_0^t f(s, x(s)) \ d_s g(t, s)) | \\
&\leq \| p(t) \| + \int_0^t | \phi(f(s, x(s))) | \ d_s(\bigvee_{z=0}^{s} g(t, z)) \\
&\leq \| p(t) \| + M \int_0^t \ d_s(\bigvee_{z=0}^{s} g(t, z)) \\
&\leq \| p(t) \| + M \bigvee_{s=0}^{t} g(t, s) \\
&\leq \| p \|_0 + M \sup_{t \in I} \bigvee_{s=0}^{t} g(t, s) \\
&\leq \| p \|_0 + MK = M_0 \ .
\end{aligned}$$

Then

$$\| Ax \|_0 = \sup_{t \in I} \| Ax(t) \| \leq M_0.$$

Hence, $Ax \in Q$ and $AQ \subset Q$ which prove that $A : Q \to Q$, and AQ is bounded in $C[I, E]$.
We need to prove now that $A : Q \to Q$ is weakly sequentially continuous. Let $\{x_n(t)\}$ be sequence in Q weakly convergent to $x(t)$ in E, since Q is closed we have $x \in Q$. Fix $t \in I$, since f satisfies (ii), then we have $f(t, x_n(t))$ converges weakly to $f(t, x(t))$. By the Lebesgue dominated convergence theorem (see assumption (iii)) for Pettis integral ([12]), we have for each $\phi \in E^*, \ s \in I$

$$\phi(\int_0^t f(s, x_n(s)) \ d_s g(t, s)) = \int_0^t \phi(f(s, x_n(s))) \ d_s g(t, s)$$

$$\to \int_0^t \phi(f(s, x(s))) \ d_s g(t, s), \ \forall \phi \in E^*, \ t \in I,$$

i.e. $\phi(Ax_n(t)) \to \phi(Ax(t)), \ \forall \ t \in I, \ Ax_n(t)$ converging weakly to $Ax(t)$ in E.
Thus, A is weakly sequentially continuous on Q.
Next we show that $AQ(t)$ is relatively weakly compact in E.
Note that Q is nonempty, closed, convex and uniformly bounded subset of $C[I, E]$ and AQ is bounded in norm. According to Propositions 1.6 and 1.7, AQ is relatively weakly compact in $C[I, E]$ implies $AQ(t)$ is relatively weakly compact in E, for each $t \in I$.
Since all conditions of Theorem 1.5 are satisfied, then the operator A has at least one fixed point $x \in Q$ and the nonlinear Stieltjes integral equation (2.5) has at least one weak solution $x \in C[I, E]$. □

3. Volterra integral equation of fractional order

In this section we show that the Volterra integral equation of fractional order

$$x(t) = p(t) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, ds,\ t \in I \tag{3.7}$$

can be considered as a special case of the Volterra-Stieltjes integral equation (2.1), where the integral is in the sense of weakly Riemann.
First, consider, as previously, that the function $g(t,s) = g : \Lambda \to R$. Moreover, we will assume that the function g satisfies the following condition

(vi') For $t_1, t_2 \in I$, $t_1 < t_2$, the function $s \to g(t_2, s) - g(t_1, s)$ is nonincreasing on $[0, t_1]$.

Now, we have the following lemmas which proved by Banaś et al. [5].

Lemma 3.1. *Under assumptions (vi$^{'}$) and (vii), for any fixed $s \in I$, the function $t \to g(t,s)$ is nonincreasing on $[s,1]$.*

Lemma 3.2. *Under assumptions (iv), (vi$^{'}$) and (vii), the function g satisfies assumption (vi).*

Consider the function g defined by

$$g(t,s) = \frac{t^\alpha - (t-s)^\alpha}{\Gamma(\alpha+1)}. \tag{3.8}$$

Now, we show that the function g satisfies assumptions (iv), (v), (vi$^{'}$) and (vii).
Clearly that the function g satisfies assumptions (iv) and (vii). Also we get

$$d_s g(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} > 0$$

for $0 \le s < t$. This implies that $s \to g(t,s)$ is increasing on $[0,t]$ for any fixed $t \in I$. Thus the function g satisfies assumption (v).
To show that g satisfies assumption (vi$^{'}$), let us fix arbitrary $t_1, t_2 \in [0,T],\ t_1 < t_2$. Then we get

$$G(s) = g(t_2, s) - g(t_1, s) = \frac{t_2^\alpha - t_1^\alpha - (t_2-s)^\alpha + (t_1-s)^\alpha}{\Gamma(\alpha+1)},$$

define on $[0, t_1]$. Thus

$$G'(s) = \frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)}\left[\frac{1}{(t_2-s)^{1-\alpha}} - \frac{1}{(t_1-s)^{1-\alpha}}\right].$$

Hence $G'(s) < 0$ for $s \in [0, t_1)$. This means that g satisfies assumption (vi$^{'}$). And the function g satisfies assumptions (iv)-(vii) in Theorem 2.5.
Hence, the equation (3.7) can be written in the form

$$x(t) = p(t) + \int_0^t f(s, x(s))\, d_s g(t,s).$$

And the equation (3.7) is a special case of the equation (2.5).
Now, we estimate the constants K, $N(\epsilon)$ used in our proof. To see this, since the function $s \to g(t,s)$ is nondecreasing on $[0,t]$ for any fixed $t \in I$. Then we have

$$\bigvee_{s=0}^{t} g(t,s) = g(t,t) - g(t,0) = g(t,t) = \frac{t^\alpha}{\Gamma(\alpha+1)},$$

and

$$\begin{aligned}\bigvee_{s=0}^{t_1} (g(t_2,s) - g(t_1,s)) &= \sum_{i=1}^{n} \mid [g(t_2,s_i) - g(t_1,s_i)] - [g(t_2,s_{i-1}) - g(t_1,s_{i-1})] \mid \\ &= \sum_{i=1}^{n} \{[g(t_2,s_{i-1}) - g(t_1,s_{i-1})] - [g(t_2,s_i) - g(t_1,s_i)]\} \\ &= g(t_1,t_1) - g(t_2,t_1) \\ &= \frac{1}{\Gamma(\alpha+1)}[t_1^\alpha - t_2^\alpha + (t_2-t_1)^\alpha].\end{aligned}$$

Thus

$$K = \sup\{\bigvee_{s=0}^{t} g(t,s) : t \in I\} = \frac{T^\alpha}{\Gamma(\alpha+1)}$$

and

$$\begin{aligned}N(\epsilon) &= \sup\Big\{ \bigvee_{s=0}^{t_1} (g(t_2,s) - g(t_1,s)) : t_1,t_2 \in I,\ t_1 < t_2,\ t_2 - t_1 \leq \epsilon \Big\} \\ &= \frac{1}{\Gamma(\alpha+1)}[t_1^\alpha - t_2^\alpha + (t_2-t_1)^\alpha].\end{aligned}$$

Since

$$\begin{aligned}\bigvee_{s=t_1}^{t_2} g(t_2,s) &= g(t_2,t_2) - g(t_2,t_1) \\ &= \frac{1}{\Gamma(\alpha+1)}[t_2^\alpha - (t_2-t_2)^\alpha - t_2^\alpha + (t_2-t_1)^\alpha] \\ &= \frac{(t_2-t_1)^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

Then

$$Q = \{x \in C[I,E] : \parallel x \parallel_0 \leq M_0\ ,$$
$$\parallel x(t_2) - x(t_1) \parallel \leq \parallel p(t_2) - p(t_1) \parallel + \frac{M}{\Gamma(\alpha+1)}[\mid t_1^\alpha - t_2^\alpha \mid + 2(t_2-t_1)^\alpha]\}.$$

Finally, we can formulate the following existence result concerning the fractional integral equation (3.7).

Theorem 3.3. *Under the assumptions (i)-(iii), the fractional integral equation (3.7) has at least one weak solution $x \in C[I,E]$.*

4. Fractional differential equations

In this section we establish existence results for the fractional differential equations (1.1)-(1.2) and (1.3)-(1.4) in the reflexive Banach space E.

4.1. Weak solution

Consider the integral equation

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, ds,\ t \in I\ , \tag{4.9}$$

where the integral is in the sense of weakly Riemann.

Lemma 4.1. *Let $\alpha \in (0,1)$. A function x is a weak solution of the fractional integral equation (4.9) if and only if x is a solution of the problem (1.1)-(1.2).*

Proof. Integrating (1.1)-(1.2) we obtain the integral equation (4.9). Operating by ${}_RD^{\alpha}$ on (4.9) we obtain the problem (1.1)-(1.2). So the equivalent between (1.1)-(1.2) and the integral equation (4.9) is proved and then the results follows from Theorem 3.3. □

4.2. Mild solution

Consider now the problem (1.3)-(1.4). According to Definitions 1.1 and 1.3, it is suitable to rewrite the problem (1.3)-(1.4) in the integral equation

$$x(t) = x_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s))\, ds,\ t \in I\ . \tag{4.10}$$

Definition 4.2. By the mild solution of the problem (1.3)-(1.4), we mean that the function $x \in C[I, E]$ which satisfies the corresponding integral equation of (1.3)-(1.4) which is (4.10).

Theorem 4.3. *If (i)-(iii) are satisfied, then the problem (1.3)-(1.4) has at least one mild solution $x \in C[I, E]$.*

It is often the case that the problem (1.3)-(1.4) does not have a differentiable solution yet does have a solution, in a mild sense.

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On the Derivative of a Polynomial with Prescribed Zeros

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Abstract: For a polynomial $p(z) = a_n \prod_{t=1}^{n}(z - z_t)$ of degree n having all its zeros in $|z| \leq K$, $K \geq 1$ it is known that

$$\max_{|z|=1} |p'(z)| \geq \frac{2}{1+K^n} \left\{ \sum_{t=1}^{n} \frac{K}{K+|z_t|} \right\} \max_{|z|=1} |p(z)| \ .$$

By assuming a possible zero of order m, $0 \leq m \leq n-4$, at $z=0$, of $p(z)$ for $n \geq k+m+1$ with integer $k \geq 3$ we have obtained a new refinement of the known result.

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Keywords and Phrases: Derivative; Polynomial; Zero of order m at 0; Refinement; Generalization.

1. Introduction and statement of results

For an arbitrary polynomial $f(z)$ let $M(f,r) = \max_{|z|=r} |f(z)|$ and $m(f,r) = \min_{|z|=r} |f(z)|$. Further let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n. Concerning the estimate of $|p'(z)|$ on $|z| \leq 1$ we have the following result due to Turán [12].

Theorem 1.1. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq 1$ then*

$$M(p',1) \geq \frac{n}{2} M(p,1).$$

The result is sharp with equality for the polynomial $p(z)$ having all its zeros on $|z| = 1$.

More generally, for the polynomial having all its zeros in $|z| \leq K$, $(K \leq 1)$, Malik [10] proved:

Theorem 1.2. *If $p(z)$ be a polynomial of degree n, having all its zeros in $|z| \leq K$, $(K \leq 1)$ then*

$$M(p',1) \geq \frac{n}{1+K} M(p,1) \ .$$

The result is sharp with equality for the polynomial $p(z) = (z+K)^n$.

And for the polynomial having all its zeros in $|z| \leq K$, $(K \geq 1)$, Govil [6] proved:

Theorem 1.3. *If $p(z)$ is a polynomial of degree n, having all its zeros in $|z| \leq K$, $(K \geq 1)$ then*

$$M(p', 1) \geq \frac{n}{1+K^n} M(p, 1)\ .$$

The result is sharp with equality for the polynomial $p(z) = z^n + K^n$.

By using the coefficients a_n, a_{n-1}, of the polynomial $p(z)$, Govil et al. [7] obtained the following refinement of Theorem 1.2.

Theorem 1.4. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, having all its zeros in $|z| \leq K$, $(K \leq 1)$ then*

$$M(p', 1) \geq n \frac{n|a_n| + |a_{n-1}|}{(1+K^2)n|a_n| + 2|a_{n-1}|} M(p, 1)\ .$$

And Aziz [1] used the moduli of all the zeros of the polynomial $p(z)$ to obtain the following refinement of Theorem 1.3.

Theorem 1.5. *If all the zeros of the polynomial $p(z) = a_n \prod_{j=1}^{n}(z - z_j)$, of degree n lie in $|z| \leq K$, $(K \geq 1)$ then*

$$M(p', 1) \geq \frac{2}{1+K^n} \Big(\sum_{j=1}^{n} \frac{K}{K + |z_j|} \Big) M(p, 1)\ .$$

The result is best possible with equality for the polynomial $p(z) = z^n + K^n$.

Later Govil [8] used certain coefficients as well as moduli of all the zeros, of the polynomial $p(z)$ to obtain the following refinement of Theorem 1.5.

Theorem 1.6. *Let $p(z) = \sum_{j=0}^{n} a_j z^j = a_n \prod_{t=1}^{n}(z - z_t)$ be a polynomial of degree n, $(n \geq 2)$, $|z_t| \leq K_t$, $1 \leq t \leq n$ and let $K = \max(K_1, K_2, \ldots, K_n) \geq 1$. Then for $n > 2$*

$$\begin{aligned} M(p', 1) \quad \geq \quad & \frac{2}{1+K^n} \Big(\sum_{t=1}^{n} \frac{K}{K + K_t} \Big) M(p, 1) \\ & + \frac{2|a_{n-1}|}{(1+K^n)} \Big(\sum_{t=1}^{n} \frac{1}{K + K_t} \Big) \Big(\frac{K^n - 1}{n} - \frac{K^{n-2} - 1}{n-2} \Big) + |a_1| \Big(1 - \frac{1}{K^2} \Big), \end{aligned}$$

and

$$\begin{aligned} M(p', 1) \geq \frac{2}{1+K^n} \Big(\sum_{t=1}^{n} \frac{K}{K + K_t} \Big) M(p, 1) + \frac{(K-1)^n}{1+K^n} |a_1| \Big(\sum_{t=1}^{n} \frac{1}{K + K_t} \Big) \\ + |a_1| \Big(1 - \frac{1}{K} \Big),\ n = 2. \end{aligned}$$

The result is best possible with equality for the polynomial $p(z) = z^n + K^n$.

Dewan et al. [3] also obtained a result similar to Theorem 1.6 for $n \geq 3$.

In this paper by assuming a possible zero of order m, $0 \leq m \leq n-4$, at $z = 0$, of $p(z)$ we have obtained a new refinement of Theorem 1.5, similar to Theorem 1.6 for $n \geq k+m+1$ with integer $k \geq 3$. More precisely we have proved

Theorem 1.7. *Let $p(z)$ be a polynomial of degree n such that*

$$p(z) = a_n \prod_{t=1}^{n}(z - z_t) = \sum_{j=0}^{n} a_j z^j, |z_t| \leq K_t, 1 \leq t \leq n$$

$$\text{and } K = \max(K_1, K_2, \ldots, K_n) \geq 1, \tag{1.1}$$

$$= z^m p_1(z),\ p_1(0) \neq 0,\ 0 \leq m \leq n-4, \tag{1.2}$$

with

$$n \geq k+m+1, (k \geq 3). \tag{1.3}$$

Then

$$M(p',1) \geq \frac{2}{1+K^{n-m}}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)M(p,1)$$

$$+\ \frac{1}{K^n}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)\frac{K^{n-m}-1}{K^{n-m}+1}m(p,K)$$

$$+\ \frac{2}{1+K^{n-m}}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)\Bigg[\frac{1}{K}.\frac{2}{\overline{n-m-1}+2}.\frac{1!|a_{n-1}|}{n-m}$$

$$\times\ \Big\{K^{n-m}-1-\binom{n-m}{1}(K-1)\Big\}$$

$$+\ \frac{1}{K^2}.\frac{2}{\overline{n-m-2}+2}.\frac{2!|a_{n-2}|}{(n-m)(n-m-1)}$$

$$\times\ \Big\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2\Big\}$$

$$+\ \frac{1}{K^3}.\frac{2}{\overline{n-m-3}+2}.\frac{3!|a_{n-3}|}{(n-m)(n-m-1)(n-m-2)}$$

$$\times\ \Big\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\binom{n-m}{3}(K-1)^3\Big\}+\ldots$$

$$+\ \frac{1}{K^{k-1}}.\frac{2}{n-m-\overline{(k-1)}+2}.\frac{(k-1)!|a_{n-(k-1)}|}{(n-m)(n-m-1)\ldots(n-m-\overline{k-2})}$$

$$\times\ \Big\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\ldots-\binom{n-m}{k-1}(K-1)^{k-1}\Big\}$$

$$+\ \frac{k!|a_{n-k}|}{K^k}\Bigg(\frac{1}{(n-m)(n-m-1)\ldots(n-m-\overline{k-1})}$$

$$\times\ \left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\ldots-\binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$-\ \frac{1}{(n-m-2)(n-m-3)\ldots(n-m-\overline{k+1})}$$

$$\times\ \left\{K^{n-m-2}-1-\binom{n-m-2}{1}(K-1)-\binom{n-m-2}{2}(K-1)^2-\ldots\right.$$

$$\left.-\binom{n-m-2}{k-1}(K-1)^{k-1}\right\}\Bigg)\Bigg]$$

$$+\ \frac{1}{K^{n-1}}\Big[\frac{2|a_1|}{n+1}(K^{n-1}-1)+\frac{2}{\overline{n-2}+2}.\frac{2!|a_2|}{n-1}\left\{K^{n-1}-1-\binom{n-1}{1}(K-1)\right\}$$

$$+\ \frac{2}{\overline{n-3}+2}.\frac{3!|a_3|}{(n-1)(n-2)}\left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2\right\}$$

$$+\ \frac{2}{\overline{n-4}+2}.\frac{4!|a_4|}{(n-1)(n-2)(n-3)}$$

$$\times\ \left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\binom{n-1}{3}(K-1)^3\right\}+\ldots$$

$$+\ \frac{2}{\overline{n-(k+m-1)}+2}.\frac{(k+m-1)!|a_{k+m-1}|}{(n-1)(n-2)\ldots(n-\overline{k+m-2})}$$

$$\times\ \left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\ldots-\binom{n-1}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$+\ (k+m)!|a_{k+m}|\Bigg(\frac{1}{(n-1)(n-2)\ldots(n-\overline{k+m-1})}$$

$$\times\ \left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\ldots-\binom{n-1}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$-\ \frac{1}{(n-3)(n-4)\ldots(n-\overline{k+m+1})}$$

$$\times\left\{K^{n-3}-1-\binom{n-3}{1}(K-1)-\binom{n-3}{2}(K-1)^2-\ldots-\binom{n-3}{k+m-2}(K-1)^{k+m-2}\right\}\Bigg)\Bigg],$$

$$n > k + m + 1, \quad (k \geq 3) \tag{1.4}$$

and

$$M(p', 1) \geq \frac{2}{1 + K^{k+1}} \Big(\sum_{t=1}^{k+m+1} \frac{K}{K + K_t} \Big) M(p, 1) + \frac{1}{K^{k+m+1}} \Big(\sum_{t=1}^{k+m+1} \frac{K}{K + K_t} \Big)$$

$$\times \frac{K^{k+1} - 1}{K^{k+1} + 1} m(p, K) + \frac{2}{1 + K^{k+1}} \Big(\sum_{t=1}^{k+m+1} \frac{K}{K + K_t} \Big)$$

$$\times \Bigg[\frac{1}{K} . \frac{2}{k + 2} . \frac{1!|a_{k+m}|}{k + 1} \Big\{ K^{k+1} - 1 - \binom{k+1}{1} (K - 1) \Big\}$$

$$+ \frac{1}{K^2} . \frac{2}{\overline{k - 1} + 2} . \frac{2!|a_{k+m-1}|}{(k + 1)k} \Big\{ K^{k+1} - 1 - \binom{k+1}{1} (K - 1) - \binom{k+1}{2} (K - 1)^2 \Big\}$$

$$+ \frac{1}{K^3} . \frac{2}{\overline{k - 2} + 2} . \frac{3!|a_{k+m-2}|}{(k + 1)k(k - 1)}$$

$$\times \Big\{ K^{k+1} - 1 - \binom{k+1}{1} (K - 1) - \binom{k+1}{2} (K - 1)^2 - \binom{k+1}{3} (K - 1)^3 \Big\} + \dots$$

$$+ \frac{1}{K^{k-1}} . \frac{2}{2 + 2} . \frac{(k - 1)!|a_{m+2}|}{(k + 1)k \dots 3}$$

$$\times \Big\{ K^{k+1} - 1 - \binom{k+1}{1} (K - 1) - \binom{k+1}{2} (K - 1)^2 - \dots - \binom{k+1}{k-1} (K - 1)^{k-1} \Big\}$$

$$+ \frac{1}{K^k} . \frac{|a_{m+1}|}{k + 1} . (K - 1)^{k+1} \Bigg] + \frac{1}{K^{k+m}} \Big[\frac{2|a_1|}{k + m + 2} (K - 1) + \frac{2}{\overline{k + m - 1} + 2} . \frac{2!|a_2|}{k + m}$$

$$\times \Big\{ K^{k+m} - 1 - \binom{k+m}{1} (K - 1) \Big\} + \frac{2}{\overline{k + m - 2} + 2} . \frac{3!|a_3|}{(k + m)(k + m - 1)}$$

$$\times \Big\{ K^{k+m} - 1 - \binom{k+m}{1} (K - 1) - \binom{k+m}{2} (K - 1)^2 \Big\}$$

$$+ \frac{2}{\overline{k + m - 3} + 2} . \frac{4!|a_4|}{(k + m)(k + m - 1)(k + m - 2)}$$

$$\times \left\{K^{k+m} - 1 - \binom{k+m}{1}(K-1) - \binom{k+m}{2}(K-1)^2 - \binom{k+m}{3}(K-1)^3\right\} + \ldots$$

$$+ \frac{2}{2+2}.\frac{(k+m-1)!|a_{k+m-1}|}{(k+m)(k+m-1)\ldots 3}$$

$$\times \left\{K^{k+m} - 1 - \binom{k+m}{1}(K-1) - \binom{k+m}{2}(K-1)^2 - \ldots - \binom{k+m}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$+ |a_{k+m}|(K-1)^{k+m}\Big], \qquad n = k+m+1, (k \geq 3). \tag{1.5}$$

Result is best possible and equality holds in (1.4) and (1.5) for $p(z) = z^n + K^n$.

Since corresponding to each of m zeros at 0, one can take

$$K_t = 0 \ ,$$

thereby implying

$$\frac{K}{K+K_t} = 1$$

and since corresponding to each of remaining $(n-m)$ zeros, we have

$$\frac{K}{K+K_t} \geq 1/2 \text{ , by } \ (1.1),$$

Theorem 1.7 gives, in particular, the following statement.

Corollary 1.8. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n, having all its zeros in $|z| \leq K$, $(K \geq 1)$ such that*

$$p(z) = z^m p_1(z), \ p_1(0) \neq 0, \ 0 \leq m \leq n-4, \tag{1.6}$$

with

$$n \geq k+m+1, (k \geq 3) \ .$$

Then

$$M(p',1) \geq \frac{n+m}{1+K^{n-m}} M(p,1) + \frac{n+m}{2K^n}.\frac{K^{n-m}-1}{K^{n-m}+1} m(p,K)$$

$$+ \frac{n+m}{1+K^{n-m}}\left[\frac{1}{K}\cdot\frac{2}{\overline{n-m-1}+2}\cdot\frac{1!|a_{n-1}|}{n-m}\left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1)\right\}\right.$$

$$+ \frac{1}{K^2}.\frac{2}{\overline{n-m-2}+2}.\frac{2!|a_{n-2}|}{(n-m)(n-m-1)}$$

$$\times\left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2\right\}$$

$$+\frac{1}{K^3}\cdot\frac{2}{\overline{n-m-3}+2}\cdot\frac{3!|a_{n-3}|}{(n-m)(n-m-1)(n-m-2)}$$

$$\times\left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\binom{n-m}{3}(K-1)^3\right\}+\ldots$$

$$+\frac{1}{K^{k-1}}.\frac{2}{\overline{n-m-(k-1)}+2}.\frac{(k-1)!|a_{n-(k-1)}|}{(n-m)(n-m-1)\ldots(n-m-\overline{k-2})}$$

$$\times\left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\ldots-\binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$+\frac{k!|a_{n-k}|}{K^k}\Bigg(\frac{1}{(n-m)(n-m-1)\ldots(n-m-\overline{k-1})}$$

$$\times\left\{K^{n-m}-1-\binom{n-m}{1}(K-1)-\binom{n-m}{2}(K-1)^2-\ldots-\binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$-\frac{1}{(n-m-2)(n-m-3)\ldots(n-m-\overline{k+1})}$$

$$\times\left\{K^{n-m-2}-1-\binom{n-m-2}{1}(K-1)-\binom{n-m-2}{2}(K-1)^2-\ldots\right.$$

$$\left.\left.\left.-\binom{n-m-2}{k-1}(K-1)^{k-1}\right\}\right)\right]$$

$$+\frac{1}{K^{n-1}}\left[\frac{2|a_1|}{n+1}(K^{n-1}-1)+\frac{2}{\overline{n-2}+2}.\frac{2!|a_2|}{n-1}\left\{K^{n-1}-1-\binom{n-1}{1}(K-1)\right\}\right.$$

$$+\frac{2}{\overline{n-3}+2}.\frac{3!|a_3|}{(n-1)(n-2)}\left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2\right\}$$

$$+\frac{2}{\overline{n-4}+2}.\frac{4!|a_4|}{(n-1)(n-2)(n-3)}$$

$$\times\left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\binom{n-1}{3}(K-1)^3\right\}+\ldots$$

$$+\frac{2}{\overline{n-(k+m-1)}+2}.\frac{(k+m-1)!|a_{k+m-1}|}{(n-1)(n-2)\ldots(n-\overline{k+m-2})}$$

$$\times\left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\ldots-\binom{n-1}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$+ (k+m)!|a_{k+m}|\Bigg(\frac{1}{(n-1)(n-2)\dots(n-\overline{k+m-1})}$$

$$\times \left\{K^{n-1}-1-\binom{n-1}{1}(K-1)-\binom{n-1}{2}(K-1)^2-\dots-\binom{n-1}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$-\ \frac{1}{(n-3)(n-4)\dots(n-\overline{k+m+1})}$$

$$\times \left\{K^{n-3}-1-\binom{n-3}{1}(K-1)-\binom{n-3}{2}(K-1)^2-\dots\right.$$

$$\left.-\binom{n-3}{k+m-2}(K-1)^{k+m-2}\right\}\Bigg)\Bigg],$$

$$n > k+m+1, \quad (k \geq 3)$$

and

$$M(p',1) \geq \frac{k+2m+1}{1+K^{k+1}}M(p,1)+\frac{k+2m+1}{2K^{k+m+1}}.\frac{K^{k+1}-1}{K^{k+1}+1}m(p,K)+\frac{k+2m+1}{1+K^{k+1}}$$

$$\times \left[\frac{1}{K}.\frac{2}{k+2}.\frac{1!|a_{k+m}|}{k+1}\left\{K^{k+1}-1-\binom{k+1}{1}(K-1)\right\}+\frac{1}{K^2}.\frac{2}{\overline{k-1}+2}.\frac{2!|a_{k+m-1}|}{(k+1)k}\right.$$

$$\times \left\{K^{k+1}-1-\binom{k+1}{1}(K-1)-\binom{k+1}{2}(K-1)^2\right\}+\frac{1}{K^3}.\frac{2}{\overline{k-2}+2}.\frac{3!|a_{k+m-2}|}{(k+1)k(k-1)}$$

$$\times \left\{K^{k+1}-1-\binom{k+1}{1}(K-1)-\binom{k+1}{2}(K-1)^2-\binom{k+1}{3}(K-1)^3\right\}+\dots$$

$$+\ \frac{1}{K^{k-1}}.\frac{2}{2+2}.\frac{(k-1)!|a_{m+2}|}{(k+1)k\dots 3}$$

$$\times \left\{K^{k+1}-1-\binom{k+1}{1}(K-1)-\binom{k+1}{2}(K-1)^2-\dots-\binom{k+1}{k-1}(K-1)^{k-1}\right\}$$

$$\left.+\ \frac{1}{K^k}.\frac{|a_{m+1}|}{k+1}(K-1)^{k+1}\right]+\frac{1}{K^{k+m}}\left[\frac{2|a_1|}{k+m+2}(K-1)+\frac{2}{\overline{k+m-1}+2}.\frac{2!|a_2|}{k+m}\right.$$

$$\times\ \left\{K^{k+m}-1-\binom{k+m}{1}(K-1)\right\}+\frac{2}{k+m-2+2}.\frac{3!|a_3|}{(k+m)(k+m-1)}$$

$$\times\ \left\{K^{k+m}-1-\binom{k+m}{1}(K-1)-\binom{k+m}{2}(K-1)^2\right\}$$

$$+\ \frac{2}{k+m-3+2}.\frac{4!|a_4|}{(k+m)(k+m-1)(k+m-2)}$$

$$\times\ \left\{K^{k+m}-1-\binom{k+m}{1}(K-1)-\binom{k+m}{2}(K-1)^2-\binom{k+m}{3}(K-1)^3\right\}+\dots$$

$$+\ \frac{2}{2+2}.\frac{(k+m-1)!|a_{k+m-1}|}{(k+m)(k+m-1)\dots 3}$$

$$\times\ \left\{K^{k+m}-1-\binom{k+m}{1}(K-1)-\binom{k+m}{2}(K-1)^2-\dots-\binom{k+m}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$\left.+\ |a_{k+m}|(K-1)^{k+m}\right],\quad n=k+m+1,\quad (k\geq 3).$$

Result is best possible with equality for the polynomial $p(z)=z^n+K^n$.

Remark 1.9. Corollary 1.8 is similar to the results ([8, Corollary] and [3, Corollary]). Further Corollary 1.8 is a refinement of Theorem 1.3 for $n\geq k+m+1$.

2. Lemmas

For the proof of Theorem 1.7 we require the following lemmas.

Lemma 2.1. *If* $p(z)=a_n\prod_{t=1}^{n}(z-z_t)$ *is a polynomial of degree* n *such that* $|z_t|\leq 1, 1\leq t\leq n$ *then*

$$M(p',1)\geq\Big(\sum_{t=1}^{n}\frac{1}{1+|z_t|}\Big)M(p,1).$$

Result is best possible with equality for the polynomial $p(z)$ *whose all zeros are positive.*

This lemma is due to Giroux et al. [5].

Lemma 2.2. *If* $p(z)=\sum_{j=0}^{n}a_jz^j$ *is a polynomial of degree* n *then*

$$M(p,R)\leq R^nM(p,1)-\frac{2|a_0|}{n+2}(R^n-1)-|a_1|\Big(\frac{R^n-1}{n}-\frac{R^{n-2}-1}{n-2}\Big),R\geq 1 \quad (2.1)$$

for $n>2$ *and*

$$M(p,R)\leq R^2M(p,1)-\frac{|a_0|}{2}(R^2-1)-\frac{|a_1|}{2}(R-1)^2,R\geq 1 \text{ for } n=2. \quad (2.2)$$

This lemma is due to Dewan et al. [3].

Lemma 2.3. *Let $p(z)$ be a polynomial of degree at most n. Then*

$$M(p', 1) \leq nM(p, 1) - \epsilon_n |p(0)|,$$

where $\epsilon_n = 2n/(n+2)$ if $n \geq 2$, where as $\epsilon_1 = 1$. The coefficient of $|p(0)|$ is the best possible for each n.

This lemma is due to Frappier et al. [4].

Lemma 2.4. *Let $p(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree $n(\geq k)$, $(k \geq 3)$. Then*

$$M(p, R) \leq R^n M(p, 1) - \frac{2|a_0|}{n+2}(R^n - 1) - \frac{2}{\overline{n-1}+2}.\frac{1!|a_1|}{n}\left\{(R^n - 1) - \binom{n}{1}(R-1)\right\}$$

$$- \frac{2}{\overline{n-2}+2}.\frac{2!|a_2|}{n(n-1)}\left\{(R^n - 1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2\right\}$$

$$- \frac{2}{\overline{n-3}+2}.\frac{3!|a_3|}{n(n-1)(n-2)}\left\{(R^n - 1) - \binom{n}{1}(R-1)\right.$$

$$\left. - \binom{n}{2}(R-1)^2 - \binom{n}{3}(R-1)^3\right\} - \ldots - \frac{2}{\overline{n-(k-2)}+2}.\frac{(k-2)!|a_{k-2}|}{n(n-1)\ldots(n-\overline{k-3})}$$

$$\times \left\{(R^n - 1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \ldots - \binom{n}{k-2}(R-1)^{k-2}\right\}$$

$$- (k-1)!|a_{k-1}|\left[\frac{1}{n(n-1)\ldots(n-\overline{k-2})}\right.$$

$$\times \left\{(R^n - 1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \ldots - \binom{n}{k-2}(R-1)^{k-2}\right\}$$

$$- \frac{1}{(n-2)(n-3)\ldots(n-k)}$$

$$\left. \times \left\{(R^{n-2} - 1) - \binom{n-2}{1}(R-1) - \binom{n-2}{2}(R-1)^2 - \ldots - \binom{n-2}{k-2}(R-1)^{k-2}\right\}\right],$$

$$R \geq 1 \text{ and } n > k, \quad (k \geq 3) \tag{2.3}$$

and

$$M(p,R) \le R^k M(p,1) - \frac{2|a_0|}{k+2}(R^k-1) - \frac{2}{\overline{k-1}+2}.\frac{1!|a_1|}{k}\left\{(R^k-1)-\binom{k}{1}(R-1)\right\}$$

$$-\frac{2}{\overline{k-2}+2}.\frac{2!|a_2|}{k(k-1)}\left\{(R^k-1)-\binom{k}{1}(R-1)-\binom{k}{2}(R-1)^2\right\}$$

$$-\frac{2}{\overline{k-3}+2}.\frac{3!|a_3|}{k(k-1)(k-2)}\left\{(R^k-1)-\binom{k}{1}(R-1)-\binom{k}{2}(R-1)^2-\binom{k}{3}(R-1)^3\right\}-\dots$$

$$-\frac{2}{2+2}.\frac{(k-2)!|a_{k-2}|}{k(k-1)\dots 4.3}\left\{(R^k-1)-\binom{k}{1}(R-1)-\binom{k}{2}(R-1)^2-\dots\right.$$

$$\left.-\binom{k}{k-2}(R-1)^{k-2}\right\} - \frac{(k-1)!|a_{k-1}|}{k!}(R-1)^k, \qquad R \ge 1 \textit{ and } n=k, (k \ge 3). \quad (2.4)$$

Lemma 2.4 is best posssible and equality holds in (2.3) and (2.4) for $p(z) = \lambda z^n$.

Proof of Lemma 2.4. We will prove inequalities (2.3) and (2.4) by mathematical induction. Accordingly for a polynomial $p(z)$ of degree

$$n > 3,$$

we have

$$\begin{aligned} |p(Re^{i\phi}) - p(e^{i\phi})| &= \left|\int_1^R p'(te^{i\phi})e^{i\phi}dt\right|, \ 0 \le \phi \le 2\pi \ , \\ &\le \int_1^R M(p',t)dt \ , \end{aligned}$$

which implies that

$$M(p,R) \le M(p,1) + \int_1^R M(p',t)dt \qquad (2.5)$$

and further as $p(z)$ is a polynomial of degree $n(>3)$, $p'(z)$ will be a polynomial of degree $(n-1)$, (>2) and therefore we can apply inequality (2.1), of Lemma 2.2, to polynomial $p'(z) = a_1 + 2a_2z + \ldots + na_nz^{n-1}$, thereby helping us to rewrite (2.5), in the form

$$M(p,R) \le M(p,1) + M(p',1)\int_1^R t^{n-1}dt - \frac{2|a_1|}{n+1}\int_1^R (t^{n-1}-1)dt - 2|a_2|$$

$$\times \int_1^R \Big(\frac{t^{n-1}-1}{n-1} - \frac{t^{n-3}-1}{n-3}\Big) dt,$$

$$\leq R^n M(p,1) - \frac{2|a_0|}{n+2}(R^n-1) - \frac{2}{n+1}.\frac{1!|a_1|}{n}\left\{(R^n-1) - \binom{n}{1}(R-1)\right\}$$

$$- 2!|a_2|\Big[\frac{1}{n(n-1)}\left\{(R^n-1) - \binom{n}{1}(R-1)\right\}$$

$$- \frac{1}{(n-2)(n-3)}\left\{(R^{n-2}-1) - \binom{n-2}{1}(R-1)\right\}\Big],$$

by Lemma 2.3.

This proves inequality (2.3) for polynomial $p(z)$ of degree $n(>k)$, with $k=3$. We can similarly prove inequality (2.4) for polynomial $p(z)$ of degree $n(=k)$, with $k=3$, by continuing in the same manner with one change:

inequality (2.2), of Lemma 2.2, to polynomial $p'(z)$ of degree $(n-1),(=2)$, instead of inequality (2.1), of Lemma 2.2, to polynomial $p'(z)$ of degree $(n-1),(>2)$.

Now we assume that inequality (2.3) is true for a polynomial $p(z)$ of degree $n(>k)$, with certain arbitrarily chosen fixed $k(\geq 3)$. Then for a polynomial $p(z)$ of degree $n(>k+1)$, with fixed $k(\geq 3)$, inequality (2.5) will obviosuly be true and as $p'(z)$ will be a polynomial of degree $(n-1)$, $(>k)$, with fixed $k(\geq 3)$, we can apply inequality (2.3) to polynomial

$$p'(z) = a_1 + 2a_2 z + 3a_3 z^2 + 4a_4 z^3 + \ldots + (k-1)a_{k-1}z^{k-2} + ka_k z^{k-1} + \ldots + na_n z^{n-1},$$

thereby helping us to rewrite (2.5) presently, in the form

$$M(p,R) \leq M(p,1) + M(p',1)\int_1^R t^{n-1}dt - \frac{2|a_1|}{n+1}\int_1^R (t^{n-1}-1)dt$$

$$- \frac{2}{\overline{n-2}+2}.\frac{1!2|a_2|}{n-1}\int_1^R \left\{(t^{n-1}-1) - \binom{n-1}{1}(t-1)\right\}dt$$

$$- \frac{2}{\overline{n-3}+2}.\frac{2!3|a_3|}{(n-1)(n-2)}\int_1^R \left\{(t^{n-1}-1) - \binom{n-1}{1}(t-1) - \binom{n-1}{2}(t-1)^2\right\}dt$$

$$- \frac{2}{\overline{n-4}+2}.\frac{3!4|a_4|}{(n-1)(n-2)(n-3)}$$

$$\times \int_1^R \left\{(t^{n-1}-1) - \binom{n-1}{1}(t-1) - \binom{n-1}{2}(t-1)^2 - \binom{n-1}{3}(t-1)^3\right\}dt - \ldots$$

$$- \frac{2}{\overline{n-1-(k-2)}+2} \cdot \frac{(k-2)!(k-1)|a_{k-1}|}{(n-1)(n-2)\ldots(n-1-\overline{k-3})}$$

$$\times \int_1^R \left\{ (t^{n-1}-1) - \binom{n-1}{1}(t-1) - \binom{n-1}{2}(t-1)^2 - \ldots - \binom{n-1}{k-2}(t-1)^{k-2} \right\} dt$$

$$- (k-1)!k|a_k| \int_1^R \left[\frac{1}{(n-1)(n-2)\ldots(n-1-\overline{k-2})} \right.$$

$$\times \left\{ (t^{n-1}-1) - \binom{n-1}{1}(t-1) - \binom{n-1}{2}(t-1)^2 - \ldots - \binom{n-1}{k-2}(t-1)^{k-2} \right\}$$

$$- \frac{1}{(n-3)(n-4)\ldots(n-1-k)}$$

$$\left. \times \left\{ (t^{n-3}-1) - \binom{n-3}{1}(t-1) - \binom{n-3}{2}(t-1)^2 - \ldots - \binom{n-3}{k-2}(t-1)^{k-2} \right\} \right] dt$$

$$\leq R^n M(p,1) - \frac{2|a_0|}{n+2}(R^n-1) - \frac{2}{\overline{n-1}+2} \cdot \frac{1!|a_1|}{n} \left\{ (R^n-1) - \binom{n}{1}(R-1) \right\}$$

$$- \frac{2}{\overline{n-2}+2} \cdot \frac{2!|a_2|}{n(n-1)} \left\{ (R^n-1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 \right\}$$

$$- \frac{2}{\overline{n-3}+2} \cdot \frac{3!|a_3|}{n(n-1)(n-2)} \left\{ (R^n-1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \binom{n}{3}(R-1)^3 \right\}$$

$$- \frac{2}{\overline{n-4}+2} \cdot \frac{4!|a_4|}{n(n-1)(n-2)(n-3)}$$

$$\times \left\{ (R^n-1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \binom{n}{3}(R-1)^3 - \binom{n}{4}(R-1)^4 \right\} - \ldots$$

$$- \frac{2}{\overline{n-(k-1)}+2} \cdot \frac{(k-1)!|a_{k-1}|}{n(n-1)\ldots(n-\overline{k-2})}$$

$$\times \left\{ (R^n-1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \ldots - \binom{n}{k-1}(R-1)^{k-1} \right\}$$

$$- k!|a_k|\Big[\frac{1}{n(n-1)\ldots(n-\overline{k-1})}$$

$$\times \left\{(R^n-1)-\binom{n}{1}(R-1)-\binom{n}{2}(R-1)^2-\ldots-\binom{n}{k-1}(R-1)^{k-1}\right\}$$

$$-\frac{1}{(n-2)(n-3)\ldots(n-\overline{k+1})}$$

$$\times \left\{(R^{n-2}-1)-\binom{n-2}{1}(R-1)-\binom{n-2}{2}(R-1)^2-\ldots-\binom{n-2}{k-1}(R-1)^{k-1}\right\}\Big], \quad (2.6)$$

by Lemma 2.3.
This proves inequality (2.3) for a polynomial $p(z)$ of degree $n(> k+1)$, with fixed $k(\geq 3)$ under the assumption that inequality (2.3) is true for a polynomial $p(z)$ of degree $n(> k)$, with certain arbitrarily chosen fixed $k(\geq 3)$. Earlier we have shown that (2.3) is true for a polynomial $p(z)$ of degree $n(> k)$, with $k = 3$. This therefore completes the proof of inequality (2.3) for a polynomial $p(z)$ of degree $n(> k)$, with $k(\geq 3)$. Again as we have proved inequality (2.3) for a polynomial $p(z)$ of degree $n(> k+1)$, with fixed $k(\geq 3)$ under the assumption that inequality (2.3) is true for a polynomial $p(z)$ of degree $n(> k)$, with certain arbitrarily chosen fixed $k(\geq 3)$, we can similarly prove inequality (2.4) for a polynomial $p(z)$ of degree $n(= k+1)$, with fixed $k(\geq 3)$ under the assumption that inequality (2.4) is true for a polynomial $p(z)$ of degree $n(= k)$, with certain arbitrarily chosen fixed $k(\geq 3)$, by continuing in the same manner with one change:

inequality (2.4) to polynomial $p'(z)$ of degree $(n-1), (= k)$,

instead of inequality (2.3) to polynomial $p'(z)$ of degree $(n-1), (> k)$,

and further we have shown earlier that inequality (2.4) is true for a polynomial $p(z)$ of degree $n(= k)$, with $k = 3$. This therefore completes the proof of inequality (2.4) for a polynomial $p(z)$ of degree $n(= k)$, with $k(\geq 3)$. This completes the proof of Lemma 2.4.

Remark 2.5. Lemma 2.4 is a refinement of well known result (see [11, Problem III 269, p. 158])

$$M(p,R) \leq R^n M(p,1), \quad R \geq 1 .$$

Remark 2.6. Lemma 2.4, along with results ([4, inequality (1.5)] and [3, Lemma 3]) suggests an inequality for $M(p,R)$ in terms of $M(p,1)$ and most of the available coefficients of the polynomial.

Lemma 2.7. *If $p(z)$ is a polynomial of degree n, having no zeros in $|z| < K_0, K_0 \geq 1$ then*

$$M(p',1) \leq \frac{n}{1+K_0}\{M(p,1) - m(p,K_0)\} .$$

The result is best possible and equality holds for $p(z) = (z+K)^n$.

This lemma is due to Govil [9].

Lemma 2.8. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* $n(\geq k+1), (k \geq 3)$*, having no zeros in* $|z| < K_0, K_0 \geq 1$*. Then*

$$M(p,R) \leq \frac{R^n + K_0}{1+K_0} M(p,1) - \frac{R^n - 1}{1+K_0} m(p,K_0) - \frac{2}{\overline{n-1}+2} \cdot \frac{1!|a_1|}{n}$$

$$\times \left\{(R^n - 1) - \binom{n}{1}(R-1)\right\} - \frac{2}{\overline{n-2}+2} \cdot \frac{2!|a_2|}{n(n-1)}$$

$$\times \left\{(R^n - 1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2\right\}$$

$$- \frac{2}{\overline{n-3}+2} \cdot \frac{3!|a_3|}{n(n-1)(n-2)}$$

$$\times \left\{(R^n - 1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \binom{n}{3}(R-1)^3\right\} - \ldots$$

$$- \frac{2}{\overline{n-(k-1)}+2} \cdot \frac{(k-1)!|a_{k-1}|}{n(n-1)\ldots(n-\overline{k-2})}$$

$$\times \left\{(R^n - 1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \ldots - \binom{n}{k-1}(R-1)^{k-1}\right\}$$

$$- k!|a_k| \Big[\frac{1}{n(n-1)\ldots(n-\overline{k-1})}$$

$$\times \left\{(R^n - 1) - \binom{n}{1}(R-1) - \binom{n}{2}(R-1)^2 - \ldots - \binom{n}{k-1}(R-1)^{k-1}\right\}$$

$$- \frac{1}{(n-2)(n-3)\ldots(n-\overline{k+1})}$$

$$\times \left\{(R^{n-2} - 1) - \binom{n-2}{1}(R-1) - \binom{n-2}{2}(R-1)^2 - \ldots - \binom{n-2}{k-1}(R-1)^{k-1}\right\},$$

$$\textit{for } R \geq 1 \quad \textit{and } n > k+1, (k \geq 3)\ , \tag{2.7}$$

and

$$M(p,R) \leq \frac{R^{k+1} + K_0}{1+K_0} M(p,1) - \frac{R^{k+1} - 1}{1+K_0} m(p,K_0) - \frac{2}{\overline{(k+1)-1}+2} \cdot \frac{1!|a_1|}{k+1}$$

$$\times \left\{(R^{k+1}-1)-\binom{k+1}{1}(R-1)\right\} - \frac{2}{(k+1)-2+2}.\frac{2!|a_2|}{(k+1)(\overline{k+1}-1)}$$

$$\times \left\{(R^{k+1}-1)-\binom{k+1}{1}(R-1)-\binom{k+1}{2}(R-1)^2\right\}$$

$$-\frac{2}{(k+1)-3+2}.\frac{3!|a_3|}{(k+1)(\overline{k+1}-1)(\overline{k+1}-2)}$$

$$\times \left\{(R^{k+1}-1)-\binom{k+1}{1}(R-1)-\binom{k+1}{2}(R-1)^2-\binom{k+1}{3}(R-1)^3\right\}-\dots$$

$$-\frac{2}{2+2}.\frac{(k-1)!|a_{k-1}|}{(k+1)(\overline{k+1}-1)\dots 4.3}$$

$$\times \left\{(R^{k+1}-1)-\binom{k+1}{1}(R-1)-\binom{k+1}{2}(R-1)^2-\dots-\binom{k+1}{k-1}(R-1)^{k-1}\right\}$$

$$-\frac{k!|a_k|}{(k+1)!}(R-1)^{k+1}\ , \quad R\geq 1 \textit{ for } n=k+1,\ (k\geq 3). \tag{2.8}$$

Proof of Lemma 2.8. As we had proved inequality (2.6) for a polynomial $p(z)$ of degree $n(>k+1)$, with fixed $k(\geq 3)$, under the assumption that ineq. (2.3) is true for a polynomial $p(z)$ of degree $n(>k)$, with certain arbitrarily chosen fixed $k(\geq 3)$, we can similarly prove inequality (2.7), (as ineq. (2.3) is now known to be true), with one change:

Lemma 2.7 instead of Lemma 2.3.

Further as we have proved ineq. (2.7), we can similarly prove ineq. (2.8), with changes:

(i) ineq. (2.4) instead of ineq. (2.3),
(ii) $n(=k)$, instead of $n(>k)$,
n(=k+1), instead of $n(>k+1)$ and
$(n-1),(=k)$, instead of $(n-1),(>k)$.

This completes the proof of Lemma 2.8.

3. Proof of Theorem 1.7

Well from (1.1) and (1.2) we can say that for $m\geq 1$, the coefficients $a_0,a_1,\dots,a_{m-1}$ will all be zero. Further $T(z)=p(Kz)$ is a polynomial of degree n, having all its zeros $z_t/K,(1\leq t\leq n)$, in $|z|\leq 1$ and therefore by Lemma 2.1

$$M(T',1)\geq \Big(\sum_{t=1}^{n}\frac{K}{K+|z_t|}\Big)M(T,1),$$

i.e.

$$KM(p',K) \geq \Big(\sum_{t=1}^{n} \frac{K}{K+K_t}\Big) M(p,K), \quad \text{by (1.1)} \ . \tag{3.1}$$

Now we first prove (1.4). As

$$p'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \ldots + n a_n z^{n-1},$$

is a polynomial of degree $(n-1)$, $(> k+m)$, by (1.4) and

$$k + m \geq 3, \quad \text{by (1.4) and (1.2)},$$

we can apply ineq. (2.3), (Lemma 2.4), to $p'(z)$, with $R = K$, thereby giving

$$M(p',K) \leq K^{n-1} M(p',1) - \frac{2|a_1|}{n+1}(K^{n-1}-1) - \frac{2}{\overline{\overline{n-2}}+2} . \frac{2!|a_2|}{n-1}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) \right\}$$

$$- \frac{2}{\overline{\overline{n-3}}+2} . \frac{3!|a_3|}{(n-1)(n-2)}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 \right\}$$

$$- \frac{2}{\overline{\overline{n-4}}+2} . \frac{4!|a_4|}{(n-1)(n-2)(n-3)}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 - \binom{n-1}{3}(K-1)^3 \right\} - \ldots$$

$$- \frac{2}{\overline{\overline{n-(k+m-1)}}+2} . \frac{(k+m-1)!|a_{k+m-1}|}{(n-1)(n-2)\ldots(n-\overline{k+m-2})}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 - \ldots - \binom{n-1}{k+m-2}(K-1)^{k+m-2} \right\}$$

$$- (k+m)!|a_{k+m}| \Bigg(\frac{1}{(n-1)(n-2)\ldots(n-\overline{k+m-1})}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 - \ldots - \binom{n-1}{k+m-2}(K-1)^{k+m-2} \right\}$$

$$- \frac{1}{(n-3)(n-4)\ldots(n-\overline{k+m+1})}$$

$$\times \left\{K^{n-3} - 1 - \binom{n-3}{1}(K-1) - \binom{n-3}{2}(K-1)^2 - \dots \right.$$

$$\left.\left. - \binom{n-3}{k+m-2}(K-1)^{k+m-2}\right\}\right). \tag{3.2}$$

It should be noted here that in (3.2), among the coefficients $a_1, a_2, \dots, a_{k+m}$, the coefficients $a_1, a_2, \dots, a_{m-1}$ will all be zero for $m > 1$, as told earlier. Further by (1.1) and (1.2) we can say that

$$p_1(z) = a_m + a_{m+1}z + \dots + a_n z^{n-m} \tag{3.3}$$

is a polynomial of degree $(n-m)$, having all its zeros in $|z| \leq K$ and therefore

$$P(z) = p_1(Kz) \tag{3.4}$$

is a polynomial of degree $(n-m)$, having all its zeros in $|z| \leq 1$, thereby implying that

$$\begin{aligned} Q(z) &= z^{n-m}\overline{P(1/\overline{z})} \\ &= z^{n-m}\overline{p_1(K/\overline{z})} \text{ (by (3.4))} \\ &= \overline{a_n}K^{n-m} + \overline{a_{n-1}}K^{n-m-1}z + \dots \\ &+ \overline{a_{m+1}}Kz^{n-m-1} + \overline{a_m}z^{n-m} \text{ (by (3.3))} \end{aligned} \tag{3.5}$$

is a polynomial of degree $(n-m)$, $(> k+1)$, $(k \geq 3)$, (by (1.3)), having no zeros in $|z| < 1$. Accordingly we can apply ineq. (2.7), (Lemma 2.8), to $Q(z)$, with $K_0 = 1$ and $R = K$, thereby giving

$$M(Q,K) \leq \frac{K^{n-m}+1}{2}M(Q,1) - \frac{K^{n-m}-1}{2}m(Q,1)$$

$$- \frac{2}{\overline{\overline{n-m-1}+2}} \cdot \frac{1!K^{n-m-1}|a_{n-1}|}{n-m}\left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1)\right\}$$

$$- \frac{2}{\overline{\overline{n-m-2}+2}} \cdot \frac{2!K^{n-m-2}|a_{n-2}|}{(n-m)(n-m-1)}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2\right\}$$

$$- \frac{2}{\overline{\overline{n-m-3}+2}} \cdot \frac{3!K^{n-m-3}|a_{n-3}|}{(n-m)(n-m-1)(n-m-2)}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \binom{n-m}{3}(K-1)^3\right\} - \dots$$

$$- \frac{2}{\overline{\overline{n-m-(k-1)}+2}} \cdot \frac{(k-1)!K^{n-m-(k-1)}|a_{n-(k-1)}|}{(n-m)(n-m-1)\dots(n-m-\overline{k-2})}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \ldots - \binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$- k!K^{n-m-k}|a_{n-k}|\Bigg[\frac{1}{(n-m)(n-m-1)\ldots(n-m-\overline{k-1})}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \ldots - \binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$- \frac{1}{(n-m-2)(n-m-3)\ldots(n-m-\overline{k+1})}$$

$$\times \left\{K^{n-m-2} - 1 - \binom{n-m-2}{1}(K-1) - \binom{n-m-2}{2}(K-1)^2 - \ldots\right.$$

$$\left. - \binom{n-m-2}{k-1}(K-1)^{k-1}\right\}\Bigg]. \tag{3.6}$$

Now by (3.5) and (1.2) we get

$$\left.\begin{array}{l} M(Q,K) = K^{n-m}M(p,1), \\ M(Q,1) = \frac{1}{K^m}M(p,K), \\ m(Q,1) = \frac{1}{K^m}m(p,K), \end{array}\right\},$$

which, on being used in (3.6), implies that

$$M(p,K) \geq \frac{2K^n}{1+K^{n-m}}M(p,1) + \frac{K^{n-m}-1}{K^{n-m}+1}m(p,K) + \frac{2K^m}{K^{n-m}+1}$$

$$\times \Bigg(\frac{2}{\overline{\overline{n-m-1}}+2}.\frac{1!K^{n-m-1}|a_{n-1}|}{n-m}\left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1)\right\}$$

$$+ \frac{2}{\overline{\overline{n-m-2}}+2}.\frac{2!K^{n-m-2}|a_{n-2}|}{(n-m)(n-m-1)}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2\right\}$$

$$+ \frac{2}{\overline{\overline{n-m-3}}+2}.\frac{3!K^{n-m-3}|a_{n-3}|}{(n-m)(n-m-1)(n-m-2)}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \binom{n-m}{3}(K-1)^3\right\} + \ldots$$

$$+ \frac{2}{\overline{n-m-(k-1)}+2} . \frac{(k-1)! K^{n-m-(k-1)} |a_{n-(k-1)}|}{(n-m)(n-m-1)\ldots(n-m-\overline{k-2})}$$

$$\times \left\{ K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \ldots - \binom{n-m}{k-1}(K-1)^{k-1} \right\}$$

$$+ k! K^{n-m-k} |a_{n-k}| \Bigg[\frac{1}{(n-m)(n-m-1)\ldots(n-m-\overline{k-1})}$$

$$\times \left\{ K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \ldots - \binom{n-m}{k-1}(K-1)^{k-1} \right\}$$

$$- \frac{1}{(n-m-2)(n-m-3)\ldots(n-m-\overline{k+1})}$$

$$\times \Bigg\{ K^{n-m-2} - 1 - \binom{n-m-2}{1}(K-1) - \binom{n-m-2}{2}(K-1)^2 - \ldots$$

$$- \binom{n-m-2}{k-1}(K-1)^{k-1} \Bigg\} \Bigg] \Bigg). \tag{3.7}$$

Finally on using inequalities (3.7) and (3.2) in ineq. (3.1) we get

$$K^n M(p', 1) - \frac{2K|a_1|}{n+1}(K^{n-1} - 1) - \frac{2K}{\overline{n-2}+2} . \frac{2!|a_2|}{n-1} \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) \right\}$$

$$- \frac{2K}{\overline{n-3}+2} . \frac{3!|a_3|}{(n-1)(n-2)} \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 \right\}$$

$$- \frac{2K}{\overline{n-4}+2} . \frac{4!|a_4|}{(n-1)(n-2)(n-3)}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 - \binom{n-1}{3}(K-1)^3 \right\} - \ldots$$

$$- \frac{2K}{\overline{n-(k+m-1)}+2} . \frac{(k+m-1)!|a_{k+m-1}|}{(n-1)(n-2)\ldots(n-\overline{k+m-2})}$$

$$\times \left\{ K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 - \ldots - \binom{n-1}{k+m-2}(K-1)^{k+m-2} \right\}$$

$$- (k+m)!|a_{k+m}|K\Bigg(\frac{1}{(n-1)(n-2)\dots(n-\overline{k+m-1})}$$

$$\times \left\{K^{n-1} - 1 - \binom{n-1}{1}(K-1) - \binom{n-1}{2}(K-1)^2 - \dots - \binom{n-1}{k+m-2}(K-1)^{k+m-2}\right\}$$

$$- \frac{1}{(n-3)(n-4)\dots(n-\overline{k+m+1})}$$

$$\times \left\{K^{n-3} - 1 - \binom{n-3}{1}(K-1) - \binom{n-3}{2}(K-1)^2 - \dots - \binom{n-3}{k+m-2}(K-1)^{k+m-2}\right\}\Bigg)$$

$$\geq \frac{2K^n}{1+K^{n-m}}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)M(p,1) + \frac{K^{n-m}-1}{K^{n-m}+1}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)m(p,K)$$

$$+ \frac{2K^m}{K^{n-m}+1}\Big(\sum_{t=1}^{n}\frac{K}{K+K_t}\Big)\Bigg(\frac{2}{\overline{n-m-1}+2}.\frac{1!K^{n-m-1}|a_{n-1}|}{n-m}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1)\right\} + \frac{2}{\overline{n-m-2}+2}.\frac{2!K^{n-m-2}|a_{n-2}|}{(n-m)(n-m-1)}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2\right\}$$

$$+ \frac{2}{\overline{n-m-3}+2}.\frac{3!K^{n-m-3}|a_{n-3}|}{(n-m)(n-m-1)(n-m-2)}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \binom{n-m}{3}(K-1)^3\right\} + \dots$$

$$+ \frac{2}{\overline{n-m-(k-1)}+2}.\frac{(k-1)!K^{n-m-(k-1)}|a_{n-(k-1)}|}{(n-m)(n-m-1)\dots(n-m-\overline{k-2})}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \dots - \binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$+ k!K^{n-m-k}|a_{n-k}|\Bigg[\frac{1}{(n-m)(n-m-1)\dots(n-m-\overline{k-1})}$$

$$\times \left\{K^{n-m} - 1 - \binom{n-m}{1}(K-1) - \binom{n-m}{2}(K-1)^2 - \ldots - \binom{n-m}{k-1}(K-1)^{k-1}\right\}$$

$$- \frac{1}{(n-m-2)(n-m-3)\ldots(n-m-\overline{k+1})}$$

$$\times \left\{K^{n-m-2} - 1 - \binom{n-m-2}{1}(K-1) - \binom{n-m-2}{2}(K-1)^2 - \ldots \right.$$

$$\left. - \binom{n-m-2}{k-1}(K-1)^{k-1}\right\}\Bigg]\Bigg)$$

and ineq. (1.4) follows.

As we have proved ineq. (1.4), we can similarly prove ineq. (1.5), with changes:
(i) $(n-1)$, $(= k+m)$, instead of $(n-1), (> k+m)$,
(ii) ineq. (2.4), (Lemma 2.4), instead of ineq. (2.3), (Lemma 2.4),
(iii) $(n-m)$, $(= k+1)$, instead of $(n-m)$, $(> k+1)$,
(iv) ineq. (2.8), (Lemma 2.8), instead of ineq. (2.7), (Lemma 2.8).
This completes the proof of Theorem 1.7.

Remark 3.1. In Theorem 1.7 possibilities

$$\left.\begin{array}{l} n-m > k+1, \\ n-m = k+1, \end{array}\right\}, k \geq 3$$

are considered and for remaining possibilities

$$\left.\begin{array}{l} n-m > k+1, \\ n-m = k+1, \end{array}\right\}, k = 2, 1, 0,$$

similar results can be obtained in a similar manner by using the results ([4, ineq. (1.5)], [3, inequalities (2.3), (2.4) and (2.5)]), Lemma 2.4, the result [2, Theorem 3], the result

$$\begin{aligned} M(p,R) &\leq \frac{R^n+1}{2}M(p,1) - \frac{R^n-1}{2}m(p,1) - |a_1|\left\{\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right\}, n > 2, \\ M(p,R) &\leq \frac{R^2+1}{2}M(p,1) - \frac{R^2-1}{2}m(p,1) - |a_1|\frac{(R-1)^2}{2}, n = 2 \end{aligned}$$

(obtained similar to the proof of Lemma 2.8 (with $K_0 = 1$), by using the results ([4, ineq. (1.5)], [3, ineq. (2.3)] and [2, Theorem 2]) and the result [3, Lemma 6]).

4. Importance of our results

Theorem 1.3 follows trivially from Corollary 1.8 by taking only first term on right hand side of inequality sign in main inequalities, (as the remaining part on right hand side of inequality sign is non-negative) and Theorem 1.5 follows trivially from Theorem 1.7 (with $K_t = |z_t|, 1 \leq t \leq n$) by taking only first term on right hand side of inequality sign in (1.4) and (1.5), (as the remaining part on right hand side of inequality sign is non-negative), $n \geq k + m + 1$, (integer $k \geq 3$ and m (= order of possible zero of $p(z)$ at $z = 0$) $\leq n - 4$).

By taking first two terms on right hand side of inequality sign in (1.4) and (1.5), one obtains the following new result (as the remaining part on right hand side of inequality sign is non-negative).

Under the hypotheses of Theorem 1.7

$$M(p', 1) \geq \frac{2}{1 + K^{n-m}} \Big(\sum_{t=1}^{n} \frac{K}{K + K_t} \Big) M(p, 1) + \frac{1}{K^n} \Big(\sum_{t=1}^{n} \frac{K}{K + K_t} \Big) \frac{K^{n-m} - 1}{K^{n-m} + 1} m(p, K).$$

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On Some Qualitative Properties of Integrable Solutions for Cauchy-type Problem of Fractional Order

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Abstract: The paper discusses the existence of solutions for Cauchy-type problem of fractional order in the space of Lebesgue integrable functions on bounded interval. Some qualitative properties of solutions are presented such as monotonicity, uniqueness and continuous dependence on the initial data. The main tools used are measure of weak (strong) noncompactness, Darbo fixed point theorem and fractional calculus.

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1. Introduction

The differential equations involving Riemann-Liouville differential operators of fractional orders are widely used in modeling several physical phenomena (see e.g. [22, 23, 25]). The problems with this approach is in that they seem to deserve an independent study of their theory parallel to the well-known theory of ordinary differential equations.
The aim of this paper is to propose the solvability of the following nonlinear Cauchy-type problem:

$$\begin{cases} D^{\alpha}\, u(t) \;=\; u(t)\cdot\int_0^1 \; k(t,\; s)\; f(s,u(s))\; ds, \quad \alpha\in(0,1), \\ \qquad\qquad\qquad u(0) \;=\; u_0, \end{cases} \tag{1.1}$$

where D^{α} is the fractional derivative (in the sense of Riemann-Liouville) of order α.
The existence result for solutions to the hybrid differential equation

$$\begin{cases} \frac{d^{\alpha}}{dt^{\alpha}}[u(t)-f(t,u(t))]=g(t,u(t)), \quad a.e.\;\; t\in[t_0,t_0+a], \\ \qquad\qquad u(0)=u_0, \end{cases}$$

has been discussed in [19] with the Riemann-Liouville fractional derivative and in [16] studied the existence of mild solutions for the above hybrid differential equation using the Caputo fractional derivative. This was extended by [24] to give approximation of solutions to Caputo fractional order hybrid differential equations.
The authors (see [15]) studied the Cauchy-type problem:

$$\begin{cases} D^\alpha u(t) = f(t, u(\varphi(t))), \\ t^{1-\alpha}\ u(t)|_{t=0} \ = \ b,\ b \ > \ 0, \end{cases}$$

where D^α is the Riemann-Liouville fractional derivative and the function $f(t, u)$ was assumed to be continuous. The existence of L_1-solution and investigated the behavior of solutions was proved in [12], where the operators were considered to be compact in L_p-spaces and this approach does not work for quadratic equations (see [10, Example 9.2]). Our approach overcomes this difficulty.

Let us recall that the quadratic integral equations were discussed in many different functions spaces (see [3, 6, 7, 8, 20]) and have numerous applications in the theories of radiative transfer, neutron transport and in the kinetic theory of gases [3, 6, 7].

Using the equivalence of the fractional Cauchy-type problem with the corresponding quadratic integral equation, we prove the existence of L_1-solution of equation (1.1).

Moreover, we discuss the monotonicity, uniqueness and Continuous dependence on the initial condition of the solution. To achieve our goal we use the technique of measure of weak (strong) noncompactness associated with Darbo fixed point theorem.

2. Notation and auxiliary facts

Let $\mathbb{R}$ be the field of real numbers, J be the interval $[0, 1]$ and $L_1(J)$ be the space of Lebesgue integrable functions (equivalence classes of functions) on a measurable subset J of $\mathbb{R}$, with the standard norm

$$\|x\|_{L_1(J)} = \int_J |x(t)|dt.$$

By $L_\infty(J)$ we denote the Banach space of essentially bounded measurable functions with the essential supremum norm (denoted by $\|x\|_{L_\infty}$). We will write L_1 and L_∞ instead of $L_1(J)$ and $L_\infty(J)$ respectively.

Definition 2.1. [2] Assume that a function $f : J \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in J$. Then to every function $x(t)$ being measurable on J we assign

$$(Fx)(t) = f(t, x(t)), \quad t \in J.$$

The operator F is called the superposition (Nemytskii) operator.

Theorem 2.1. *[2] Let f satisfies the Carathéodory conditions. The operator F maps continuously the space L_1 into itself if and only if*

$$|f(t,x)| \leq a(t) + q \cdot |x|,$$

for all $t \in J$ and $x \in \mathbb{R}$, where $a \in L_1$ and $q \geq 0$.

Let $S = S(J)$ denote the set of measurable (in Lebesgue sense) functions on J and let *meas* stands for the Lebesgue measure in $\mathbb{R}$. Identifying the functions equal almost everywhere the set S furnished with the metric

$$d(x,y) = \inf_{a>0}[a + meas\{s : |x(s) - y(s)| \geq a\}],$$

becomes a complete metric space. Moreover, the convergence in measure on J is equivalent to the convergence with respect to the metric d (Proposition 2.14 in [26]). The compactness in such a space is called a "compactness in measure".

Theorem 2.2. *[20] Let X be a bounded subset of L_1 and suppose that there is a family of measurable subsets $(\Omega_c)_{0\leq c\leq 1}$ of the interval J such that $meas\Omega_c = c$ for every $c \in J$and for $x \in X$*

$$x(t_1) \geq x(t_2), \quad (t_1 \in \Omega_c, \quad t_2 \notin \Omega_c).$$

Then X is compact in measure subset of L_1.

Theorem 2.3. *[18, Theorem 6.2] The linear integral operator $K : L_1 \to L_1$ given by a formula $(Ku)(t) = \int_J k(t,s)u(s)\, ds$ preserve the monotonicity of functions iff*

$$\int_0^l k(t_1, s)\, ds \geq \int_0^l k(t_2, s)\, ds$$

for $t_1 < t_2$, $t_1, t_2 \in J$ and for any $l \in J$.

Assume that $(E, \|\cdot\|)$ is an arbitrary Banach space with zero element θ. Denote by $B(x,r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. Denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$, $\mathcal{N}_E^W$ its subfamilies consisting of all relatively compact and weakly relatively compact sets, respectively. The symbols $\bar{X}, \bar{X}^W$ stand for the closure and the weak closure of a set X, respectively and $convX$ will denote the convex closure of X.

Definition 2.2. [4] A mapping $\mu : \mathcal{M}_E \to [0, \infty)$ is called a regular measure of noncompactness in E if the following conditions hold:

(i) $\mu(X) = 0 \Leftrightarrow X \in \mathcal{N}_E$.

(ii) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(iii) $\mu(\bar{X}) = \mu(convX) = \mu(X)$.

(iv) $\mu(\lambda X) = |\lambda|\,\mu(X),\ for\ \lambda \in \mathbb{R}$.

(v) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.

(vi) $\mu(X \bigcup Y) = \max\{\mu(X),\ \mu(Y)\}$.

(vii) If X_n is a sequence of nonempty, bounded, closed subsets of E, $X_n = \bar{X_n}$ such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \ldots$, and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

An example of such a mapping is the following:

Definition 2.3. [4] Let X be a nonempty and bounded subset of E. The Hausdorff measure of noncompactness $\chi(X)$ is defined as

$$\chi(X) = inf\{r > 0 : \text{ there exists a finite subset Y of E such that } x \subset Y + B_r\}.$$

Definition 2.4. [4] A mapping $\mu : \mathcal{M}_E \to [0,\ \infty)$ is called a regular measure of weak noncompactness in E if it satisfies conditions (ii)-(vi) of Definition 2.2 and the following two conditions (being counterparts of (i) and (vii)) hold:

(i') $\mu(X) = 0 \Leftrightarrow X \in \mathcal{N}_E^W$.

(vii') If X_n is a sequence of nonempty, bounded, closed subsets of E, $X_n = \bar{X_n}^W$ such that $X_{n+1} \subset X_n$, $n = 1, 2, 3, \ldots$, and $\lim_{n\to\infty} \mu(X_n) = 0$, then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

Another regular measure was defined in the space L_1 (cf. [5]). For any $\varepsilon > 0$, let c be a measure of equiintegrability of the set X [2, p. 39] i.e.

$$c(X) = \lim_{\varepsilon\to 0}\{\sup_{x\in X}\{\sup[\int_D |x(t)|\, dt,\ D \subset J,\ measD \leq \varepsilon]\}\}.$$

It forms a regular measure of noncompactness if restricted to the family of subsets being compact in measure (cf. [14]).

Theorem 2.4. *[4] Let Q be a nonempty, bounded, closed, and convex subset of E and let $H : Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists $\gamma \in [0, 1)$ such that*

$$\mu(H(X)) \leq \gamma\mu(X),$$

for any nonempty subset X of E. Then H has at least one fixed point in the set Q.

Next, we introduce a short note about fractional calculus theory.

Definition 2.5. [17] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}ds$$

provided that the right side is pointwise defined on $(0, \infty)$, where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.6. [17] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds$$

provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.1. *[13, 17]. Let $f \in L_1$ and let $\alpha \in (0, 1)$, then*

(a) $D^\alpha I^\alpha f(t) = f(t)$.

(b) $I^\alpha D^\alpha f(t) = f(t) - f(0)$.

(c) The operator I^α maps L_1 into itself continuously.

(d) The operator I^α maps the monotonic nondecreasing function into functions of the same type.

3. Main result

First we need to prove the equivalence of (1.1) with the corresponding quadratic integral equation:

$$u(t) = u_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(u(s) \cdot \int_0^1 k(s,\ v) f(v, u(v)) dv \right) ds, \quad t \in (0, 1). \tag{3.1}$$

Indeed, let $u(t)$ be a solution of (1.1). Applying the operator I^α to both sides of (1.1). By Lemma 2.1, we have

$$I^\alpha D^\alpha u(t) \ = u(t) - u(0) = \ I^\alpha \left(u(t) \cdot \int_0^1 \ k(t,\ s)\ f(s, u(s))\ ds \right),$$

where $u(0) = u_0$, we have

$$u(t) = \ u_0 \ + \ \int_0^t \frac{(t \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \ \left(u(s) \cdot \int_0^1 \ k(s,\ v)\ f(v, u(v))\ dv \ \right) ds.$$

Thus equation (3.1) holds.

Conversely, let $u(t)$ be a solution of equation (3.1). Then applying D^α on both sides of (3.1), we obtain (1.1). Finally, put $t = 0$ in equation (3.1), we get $u(0) = u_0$. Then problem (1.1) and equation (3.1) are equivalent to each other.

3.1. Existence of monotonic integrable solution

Rewrite (3.1) as

$$u = Tu,$$

where

$$Tu(t) = u_0 \; + \; I^{\alpha}(Au)(t), \; (Au)(t) = u(t) \cdot (KFu)(t),$$

$(Ku)(t) = \int_0^1 k(t,s)u(s)ds, \; Fu = f(t,u)$, and I^{α} is as in Definition 2.6.

To solve equation (3.1) it is necessary to find a fixed point of the operator T. For facilitating our discussion, we shall treat (3.1) under the following assumptions listed below:

(i) $f : J \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions and $f(t,u) \geq 0$ for $u \geq 0$. Moreover, f is assumed to be nondecreasing with respect to both variables t and u separately.

(ii) There is a positive function $a \in L_1$ and a constant $q \geq 0$ such that

$$|f(t,u)| \; \leq \; a(t) \; + \; q \; |u|,$$

for all $t \in J$ and $u \in \mathbb{R}$.

(iii) $k(t,s) : J \times J \to \mathbb{R}$ is measurable with respect to both variables. The linear integral operator K associated with the kernel k maps L_1 into L_∞ and is continuous.

(iv) For any $0 < l < 1$ the following condition holds true

$$t_1 < t_2 \quad \Rightarrow \quad \int_0^l k(t_1,s)\,ds \geq \int_0^l k(t_2,s)\,ds.$$

(v) Let $r > 0$ be such that $\|K\|_{L_\infty} \; (\; \|a\|_{L_1} + q \cdot r) < \Gamma(\alpha+1)$.

Theorem 3.1. *Let assumptions (i)-(v) be satisfied, then the Cauchy-type problem* (1.1) *has at least one solution in* L_1 *a.e. nondecreasing on* J.

Proof. First of all observe that by assumptions (i), (ii) and Theorem 2.1 we have that the superposition operator F maps continuously L_1 into itself. From assumption (iii) the operator (KF) maps L_1 into L_∞. From the Hölder inequality the operator A maps L_1 into itself and is continuous. Finally, since the operator I^{α} maps L_1 into itself continuously, then we can deduce that the operator T maps L_1 into itself and is continuous.

Thus for $u \in L_1$, we have

$$\begin{aligned}
\|Tu\|_{L_1} &= \int_0^1 |(Tu)(t)|dt \leq \int_0^1 u_0 \ dt \ + \ \int_0^1 I^\alpha(Au)(t)dt \\
&\leq u_0 \cdot t \Big|_0^1 + \int_0^1 \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(u(s) \cdot \int_0^1 \ k(s, \ v) \ f(v, u(v)) \ dv \right) ds \right| dt \\
&\leq u_0 + \int_0^1 \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(u(s) \cdot \int_0^1 \ k(s, \ v) \ (a(v) + q|u(v)|) \ dv \right) ds \right| dt \\
&\leq u_0 + \int_0^1 \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) \cdot \|K\|_{L_\infty} \ (\|a\|_{L_1} + q\|u\|_{L_1}) ds \right| dt \\
&\leq u_0 + \|K\|_{L_\infty} \ (\|a\|_{L_1} + q\|u\|_{L_1}) \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \int_0^1 |u(s)| ds \\
&\leq u_0 + \|K\|_{L_\infty} \ (\|a\|_{L_1} + q\|u\|_{L_1}) \frac{t^\alpha}{\alpha\Gamma(\alpha)} \Big|_0^1 \cdot \|u\|_{L_1} \\
&= u_0 + \frac{\|K\|_{L_\infty}}{\Gamma(\alpha+1)} \ (\|a\|_{L_1} + q\|u\|_{L_1}) \cdot \|u\|_{L_1},
\end{aligned}$$

so the function Tu is bounded in J. This allows us to infer that the operator T transforms L_1 into itself. Moreover, this estimate the following

$$\|Tu\|_{L_1} \leq \ u_0 + \frac{\|K\|_{L_\infty}}{\Gamma(\alpha+1)} \ (\|a\|_{L_1} + q \cdot r) \cdot r \leq \ r.$$

As a domain for the operator T we will consider the ball B_r, where r is the positive solution of the equation

$$u_0 + \frac{\|K\|_{L_\infty}}{\Gamma(\alpha+1)} \ (\|a\|_{L_1} + q \cdot r) \cdot r = \ r.$$

Let us remark, that the above inequality is of the form $\hat{a} + (\hat{b} + \hat{v}r)\hat{c}r \leq r$ with $\hat{a}, \hat{b}, \hat{c}, \hat{v} > 0$. Then $\hat{v}\hat{c} > 0$ and by assumption (v), we have that $\hat{b}\hat{c} - 1 < 0$ and that the discriminant is positive, then Viète's formulas imply that the quadratic equation has two positive solutions $r_1 < r_2$. So there exists a positive number $r > 0$ satisfying this inequality such that T maps the ball B_r into itself and is continuous.

Further, let Q_r is a subset of B_r which has the functions a.e. nondecreasing on J. This set is nonempty, bounded (by r), convex and closed in L_1. In view of Theorem 2.2 the set Q_r is compact in measure.

Now, we will show that T preserve the monotonicity of functions. Take $u \in Q_r$, then $u(t)$ is a.e. nondecreasing on J and consequently F is also of the same type by virtue of the assumption (i). Further, $Ku(t)$ is a.e. nondecreasing on J due to assumption (iv). Since the pointwise product of a.e. monotone functions is still of the same type, the operator $A = u \cdot (KFu)$ is a.e. nondecreasing on J. Moreover, the operator I^α maps a.e. nondecreasing functions into functions of the same type (thanks to Lemma 2.1). Thus we can deduce that Tu is also a.e. nondecreasing on J. Then T maps continuously Q_r into Q_r.

From now we will assume that X is a nonempty subset of Q_r and the constant $\varepsilon > 0$ is arbitrary, but fixed. Then for an arbitrary $u \in X$ and for a set $D \subset J$, $measD \leq \varepsilon$ we obtain

$$\begin{aligned}
\|Tu\|_{L_1(D)} &= \int_D |Tu(t)|dt \leq \int_J u_0 \cdot \chi_D(t)\ dt\ +\ \int_D I^\alpha (Au)(t)dt \\
&\leq \|u_0 \cdot \chi_D\|_{L_1} \\
&\quad + \int_D \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(u(s) \cdot \int_0^1 k(s,\ v)\ f(v, u(v))\ dv \right) ds \right| dt \\
&\leq \|u_0 \cdot \chi_D\|_{L_1} \\
&\quad + \int_D \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (u(s) \cdot \|K\|_{L_\infty} [\ \|a\|_{L_1} + q\|u\|_{L_1}])\, ds \right| dt \\
&\leq \|u_0 \cdot \chi_D\|_{L_1}\ +\ \|K\|_{L_\infty} (\ \|a\|_{L_1} + q \cdot r) \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \cdot \int_D |u(s)|ds \\
&\leq \|u_0 \cdot \chi_D\|_{L_1}\ +\ \frac{\|K\|_{L_\infty}(\ \|a\|_{L_1} + q \cdot r)}{\Gamma(\alpha+1)} \cdot \int_D |u(s)|ds.
\end{aligned}$$

Hence, taking into account the equality

$$\lim_{\varepsilon \to 0} \{\sup[\|u_0 \cdot \chi_D\|_{L_1}\ : D \subset J,\ measD \leq \varepsilon]\} = 0,$$

and by the definition of $c(X)$, we get

$$c(TX) \leq \frac{\|K\|_{L_\infty}(\ \|a\|_{L_1} + q \cdot r)}{\Gamma(\alpha+1)} \cdot c(X).$$

Recall that $\|K\|_{L_\infty}\ (\ \|a\|_{L_1} + q \cdot r) < \Gamma(\alpha+1)$ and the inequality obtained above with the properties of the operator T and since the set Q_r is compact in measure we are able to apply Theorem 2.4 which completes the proof. □

3.2. Uniqueness of the solution

We will discuss the uniqueness of solutions in the following theorem:

Theorem 3.2. *Let assumptions of Theorem 3.1 be satisfied, but instead of assumptions (ii) and (v) consider the following condition:*

(vi) Assume that, there is a constant $q \geq 0$ such that for a.e. $t \in J$

$$|f(t,u) - f(t,v)| \leq q|u-v|, \text{ and } |f(t,0)| \leq a(t),$$

where $u, v \in \mathbb{R}$ and $a \in L_1$.
Moreover, let

$$\left(\frac{\|K\|_{L_\infty}\ (\|a\|_{L_1} + 2q \cdot r)}{\Gamma(\alpha+1)} \right) < 1.$$

Then the Cauchy-type problem (1.1) has an unique solution in $L_1(J)$.

Proof. From assumption (vi), we have

$$\begin{aligned} ||f(t,u)| - |f(t,0)|| &\leq |f(t,u) - f(t,0)| \leq q|u| \\ \Rightarrow |f(t,u)| &\leq |f(t,0)| + q|u| \leq a(t) + q|u|, \end{aligned}$$

which imply that assumption (ii) and (v) of Theorem 1.1 are satisfied.

For the uniqueness solution of (1.1), let $x(t)$ and $y(t)$ be any two solutions of (3.1) in B_r, then we have

$$\begin{aligned} |x(t) - y(t)| &\leq \left| I^{\alpha}\left(x(t)\cdot\int_0^1 k(t, v)\ f(v,x(v))\ dv\right)\right. \\ &\quad \left. - I^{\alpha}\left(y(t)\cdot\int_0^1 k(t, v)\ f(v,y(v))\ dv\right)\right| \\ &\leq \left| I^{\alpha}\left(x(t)\cdot\int_0^1 k(t, v)\ f(v,x(v))\ dv\right)\right. \\ &\quad \left. - I^{\alpha}\left(x(t)\cdot\int_0^1 k(t, v)\ f(v,y(v))\ dv\right)\right| \\ &\quad + \left| I^{\alpha}\left(x(t)\cdot\int_0^1 k(t, v)\ f(v,y(v))\ dv\right)\right. \\ &\quad \left. - I^{\alpha}\left(y(t)\cdot\int_0^1 k(t, v)\ f(v,y(v))\ dv\right)\right| \\ &\leq I^{\alpha}\left(|x(t)|\cdot\int_0^1 |k(t, v)|\ |f(v,x(v)) - f(v,y(v))|\ dv\right) \\ &\quad + I^{\alpha}\left(|x(t)-y(t)|\cdot\int_0^1 |k(t, v)|\ |f(v,y(v))|\ dv\right) \\ &\leq I^{\alpha}\left(|x(t)|\cdot\int_0^1 |k(t, v)|\ q|x(v)-y(v)|\ dv\right) \\ &\quad + I^{\alpha}\left(|x(t)-y(t)|\cdot\int_0^1 |k(t, v)|\ [a(v)+q|y(v))|\ dv\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \|x - y\|_{L_1} &\leq \int_0^1 \left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(|x(s)|\cdot\int_0^1 |k(s, v)|\ q|x(v)-y(v)|\ dv\right) ds\right\} dt \\ &\quad + \int_0^1 \left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right. \\ &\quad \left.\left(|x(s)-y(s)|\cdot\int_0^1 |k(s, v)|\ [a(v)+q|y(v)|\ dv\right) ds\right\} dt \end{aligned}$$

$$\leq \frac{\|x\|_{L_1} \cdot \ q\|K\|_{L_\infty}}{\Gamma(\alpha+1)} \cdot \|x-y\|_{L_1} + \frac{\|K\|_{L_\infty} \ (\|a\|_{L_1} + q\|y\|_{L_1})}{\Gamma(\alpha+1)} \cdot \|x-y\|_{L_1}$$

$$\leq \left(\frac{q \cdot r \cdot \|K\|_\infty + \|K\|_{L_\infty} \ (\|a\|_{L_1} + q \cdot r)}{\Gamma(\alpha+1)} \right) \cdot \|x-y\|_{L_1}.$$

The above inequality yields

$$\left(1 - \frac{\|K\|_{L_\infty} \ (\|a\|_{L_1} + 2q \cdot r)}{\Gamma(\alpha+1)} \right) \cdot \|x-y\|_{L_1} \leq 0,$$

which implies that

$$\|x-y\|_{L_1} \ = \ 0 \ \Rightarrow \ x \ = \ y.$$

This complete the proof. □

3.3. Continuous dependence on the initial condition

In the present section sufficient conditions are obtained under which the solution $u(t, 0, u_0)$ of problem (1.1) depends continuously on the initial conditions.

Definition 3.1. [11] We say that the solution $u(t, 0, u_0)$ of problem (1.1) depends continuously on the initial conditions for $t \in J$, if for every two positive numbers ϵ and η there exists a number $\delta = \delta(\epsilon, \eta) > 0$ such that if $|u_0 - \tilde{u_0}| < \delta$ then $\|u(t, 0, u_0) - \tilde{u}(t, 0, \tilde{u_0})\|_{L_1} < \epsilon$ for $t \in J$ and $|t| > \eta$.

Theorem 3.3. *Let assumptions of Theorem 3.2 be satisfied, then the solution of the Cauchy-type problem (1.1) depends continuously on the initial condition in J.*

Proof. According to Theorem 3.2, the problem (1.1) has an unique solution. We will prove that this solution depends continuously on the initial condition in J.

Let $u(t)$ be a solution of

$$u(t) \ = \ u_0 \ + \ \int_0^t \frac{(t \ - \ s)^{\alpha-1}}{\Gamma(\alpha)} \left(u(s) \cdot \int_0^1 \ k(s, \ \upsilon) \ f(\upsilon, u(\upsilon)) \ d\upsilon \ \right) ds,$$

and let $\tilde{u}(t)$ be a solution of the above equation such that $\tilde{u}(0) = \tilde{u_0}$, then

$$\begin{aligned}
|u(t) - \tilde{u}(t)| &= |u_0 - \tilde{u_0}| \\
&+ \left| I^\alpha \left(u(t) \cdot \int_0^1 k(t,\ \upsilon)\ f(\upsilon, u(\upsilon))\ d\upsilon \right) ds \right. \\
&\left. - I^\alpha \left(\tilde{u}(t) \cdot \int_0^1 k(t,\ \upsilon)\ f(\upsilon, \tilde{u}(\upsilon))\ d\upsilon \right) ds \right| \\
&\leq |u_0 - \tilde{u_0}| + I^\alpha \left(|u(t)| \cdot \int_0^1 |k(t,\ \upsilon)|\ |f(\upsilon, u(\upsilon)) - f(\upsilon, \tilde{u}(\upsilon))|\ d\upsilon \right) \\
&+ I^\alpha \left(|u(t) - \tilde{u}(t)| \cdot \int_0^1 |k(t,\ \upsilon)|\ |f(\upsilon, \tilde{u}(\upsilon))|\ d\upsilon \right) \\
&\leq |u_0 - \tilde{u_0}| + I^\alpha \left(|u(t)| \cdot \int_0^1 |k(t,\ \upsilon)|\ q|u(\upsilon) - \tilde{u}(\upsilon)|\ d\upsilon \right) \\
&+ I^\alpha \left(|u(t) - \tilde{u}(t)| \cdot \int_0^1 |k(t,\ \upsilon)|\ [|a(\upsilon) + q|\tilde{u}(\upsilon)|]\ d\upsilon \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|u\ -\ \tilde{u}\|_{L_1} &\leq \int_0^1 |u_0 - \tilde{u_0}| dt \\
&+ \int_0^1 \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \left(|u(s)| \cdot \int_0^1 |k(s,\ \upsilon)|\ q|u(\upsilon) - \tilde{u}(\upsilon)|\ d\upsilon \right) ds \right\} dt \\
&+ \int_0^1 \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right. \\
&\left. \left(|u(s) - \tilde{u}(s)| \cdot \int_0^1 |k(s,\ \upsilon)|\ [|a(\upsilon) + q|\tilde{u}(\upsilon)|]\ d\upsilon \right) ds \right\} dt \\
&\leq |u_0 - \tilde{u_0}| + \left(\frac{q \cdot r \cdot \|K\|_\infty + \|K\|_{L_\infty}\ (\|a\|_{L_1} + q \cdot r)}{\Gamma(\alpha+1)} \right) \cdot \|u - \tilde{u}\|_{L_1}.
\end{aligned}$$

Then we get

$$\|u\ -\ \tilde{u}\|_{L_1} \quad \leq \quad \left(1 - \frac{\|K\|_{L_\infty}\ (\|a\|_{L_1} + 2q \cdot r)}{\Gamma(\alpha+1)} \right)^{-1} \cdot |u_0 - \tilde{u_0}|.$$

Therefore, if $|u_0 - \tilde{u_0}| < \delta(\epsilon)$, then $\|u - \tilde{u}\|_{L_1} < \epsilon$, where

$$\delta(\epsilon) = \epsilon \cdot \frac{\Gamma(\alpha+1) - \|K\|_{L_\infty}\ (\|a\|_{L_1} + 2q \cdot r)}{\Gamma(\alpha+1)}.$$

Now from the equivalence of (1.1) and (3.1), we get that the solution of the Cauchy-type problem (1.1) depends continuously on the initial condition in J. □

3.4. Remarks

We need to stress on some aspects of obtained results. First of all we can observe, that our solutions are not necessarily continuous as in almost all previously investigated cases (cf. [1, 21]). So we need not to assume, that the Hammerstein operator transforms the space $C(I)$ into itself. Our solutions belong to the space L_1, for the examples and conditions related to Hammerstein operators in L_1 we refer the readers to [20, 27].

We need to emphasize that the following Cauchy-type problem is also strictly related to quadratic equations (cf. [8])

$$\left(\frac{x(t)-g(t)}{f_1(t,x)}\right)' = f_2(t,x(t)), \quad x(0)=0,$$

where $f_1 : I \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$.

It can be easily checked that under some typical assumptions this problem is equivalent to the integral equation [9]

$$x(t) \;=\; g(t) \;+\; f_1(t,x(t)) \cdot \left(\int_0^t f_2(s,x(s))\,ds - \frac{g(0)}{f_1(0,0)}\right).$$

Nevertheless, when we are looking for continuous solutions for integral equation, for differential one we obtain classical solutions, i.e. x is continuously differentiable. In the case presented above we investigate Carathéodory solutions for the Cauchy problem.

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Approximation by Szász Type Operators Including Sheffer Polynomials

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Abstract: In present article, we discuss voronowskaya type theorem, weighted approximation in terms of weighted modulus of continuity for Szász type operators using Sheffer polynomials. Lastly, we investigate statistical approximation for these sequences.

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Keywords and Phrases: Szász operators; Sheffer Polynomials; Voronovskaya.

1. Introduction

First, we recall n^{th} Bernstein operators due to Bernstein [1] defined as follows

$$B_n(f;x)=\sum_{k=0}^{n}p_{n,k}(x)f\left(\frac{k}{n}\right), \tag{1.1}$$

where $p_{n,k}(x)=\binom{n}{k}x^k(1-x)^{n-k}$, $f\in C[0,1]$ and $0\le x\le 1$. The purpose of this probabilistic method was to prove Weierstass approximation theorem more elegantly. In 1950, Szász [6] generalized operators given by (1.1) for unbounded interval on the space of continuous functions defined on $(0,\infty)$ as

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \qquad \forall x\in(0,\infty), \qquad n\in\mathbb{N}. \tag{1.2}$$

A new type of generalization of Szász-Mirakjan operators which involves Appell polynomials was given by Jakimovski and Leviatan [4] as follows

$$P_n(f;x)=\frac{e^{-nx}}{A(1)}\sum_{k=0}^{\infty}p_k(nx)f\left(\frac{k}{n}\right).$$

In above relation p_k are Appell polynomials defined by the generating functions

$$A(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k,$$

where $A(z) = \sum_{k=0}^{\infty} a_k z^k (a_0 = 0)$ is an analytic function in the disc $|z| < R$ $(R > 1)$ and $A(z) \neq 0$. A more generalized form of Szász operators including Sheffer polynomials was given by Ismail [3]

$$T_n(f;x) = \frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right). \tag{1.3}$$

In above relation p_k are Sheffer polynomials given by the generating functions

$$A(u)e^{xH(u)} = \sum_{k=0}^{\infty} p_k(x)u^k, \tag{1.4}$$

where

$$A(z) = \sum_{k=0}^{\infty} a_k z^k \qquad (a_0 \neq 0)$$

$$H(z) = \sum_{k=0}^{\infty} h_k z^k \qquad (h_1 \neq 0) \tag{1.5}$$

be analytic functions in the disc $\mid z \mid< R$ $(R > 1)$. Under the following restrictions:

(i) for $x \in [0, \infty)$ and $k \in N \cup 0$, $p_k(x) \geq 0$,

(ii) $A(1) \neq 0$ and $H'(1) = 1$,

(iii) relation (1.4) is valid for $\mid u \mid< R$ and the power series given by (1.5) converges for $\mid z \mid< R$, $R > 1$. Moreover, Ismail introduced the Kantorovich form of the operator (1.3) as

$$T_n^*(f;x) = n\frac{e^{-nxH(1)}}{A(1)} \sum_{k=0}^{\infty} p_k(nx) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(s)ds. \tag{1.6}$$

Recently, Sucu and Ertan Ibikli [7] proved results on rate of convergence using modulus of continuity for (1.3) and (1.6). Motivated by the above development, we prove weighted approximation, statistical approximation and Voronovskaya type result for T_n in the present paper.

Various investigators such as Gairola et al. [9], Singh et al. [10], Mishra et al. [16-21], Gandhi et al. [22] and the references therein, have discussed the approximation properties of various linear positive operators in this direction.

2. Some properties of the operator T_n

We recall following lemmas due to Sezgin et al. [7]:

Lemma 2.1. *Let $e_i = t^i$, $i = 0, 1, 2$, $x \in [0, \infty)$, we have*

$$\begin{aligned} T_n(e_0; x) &= 1, \\ T_n(e_1; x) &= x + \frac{A'(1)}{nA(1)}, \\ T_n(e_2; x) &= x^2 + \left(\frac{2A'(1)}{A(1)} + H''(1) + 1\right)\frac{x}{n} + \frac{A'(1) + A''(1)}{n^2 A(1)}. \end{aligned}$$

Lemma 2.2. *Let $\psi_x^i(t) = (t - x)^i, i = 0, 1, 2$, for $x \geq 0$ and $n \in N$ we have*

$$\begin{aligned} T_n(\psi_x^0(t); x) &= 1, \\ T_n(\psi_x^1(t); x) &= \frac{A'(1)}{nA(1)}, \\ T_n(\psi_x^2(t); x) &= \left(\frac{H''(1) + 1}{n}\right) + \frac{A'(1) + A''(1)}{n^2 A(1)}. \end{aligned}$$

Next we prove

Lemma 2.3. *For $x \geq 0$, we have*

$$\begin{aligned} T_n(e_3; x) &= x^3 + \left(3 + \frac{3A'(1)}{A(1)} + 3H''(1)\right)\frac{x^2}{n} \\ &+ \left(\frac{2 + 3A''(1)}{A(1)} + \frac{6A'(1)}{A(1)} + \frac{3A'(1)H''(1)}{A(1)} + H''(1) + H'''(1)\right)\frac{x}{n^2} \\ &+ \frac{2A'(1) + 3A''(1) + A'''(1)}{n^3 A(1)}, \\ T_n(e_4; x) &= x^4 + \left(6 + \frac{4A'(1)}{A(1)} + 6H''(1)\right)\frac{x^3}{n} \\ &+ \left(11 + \frac{6A''(1)}{A(1)} + \frac{18A'(1)}{A(1)} + 18H''(1) + \frac{9A'(1)H''(1)}{A(1)} + 3(H''(1))^2 \right. \\ &+ \left. 4H'''(1)\right)\frac{x^2}{n^2} + \left(6 + \frac{4A'''(1)}{A(1)} + \frac{18A''(1)}{A(1)} + \frac{22A'(1)}{A(1)} + \frac{6A''(1)H''(1)}{A(1)} \right. \\ &+ \left. \frac{18A'(1)H''(1)}{A(1)} + \frac{4A'(1)H'''(1)}{A(1)} + 6H'''(1) + 11H''(1) + H''''(1)\right)\frac{x}{n^3} \\ &+ \frac{6A'(1) + 11A''(1) + A''''(1)}{A(1)}. \end{aligned}$$

Proof. From the generating functions of Sheffer polynomials, we obtain

$$\begin{aligned}\sum_{K=0}^{\infty} K^3 P_K(nx) &= [(2A'(1)+3A''(1)+A'''(1)) \\ &+ nx(3A''(1)+6A'(1)+3A'(1)H''(1) \\ &+ 3A(1)H''(1)+2A(1)+A(1)H'''(1))+n^2 2x^2(3A(1)+3A'(1) \\ &+ 3A(1)H''(1))+n^3x^3A(1)]e^n xH(1), \\ \sum_{K=0}^{\infty} K^4 P_K(nx) &= [(6A'(1)+11A''(1)+6A'''(1) \\ &+ A''''(1))+nx(4A'''(1)+18A''(1) \\ &+ 22A'(1)+6A''(1)H''(1)+18A'(1)H''(1)+4A'(1)H'''(1) \\ &+ 6A(1)H'''(1)+\ 1A(1)H''(1)+6A(1)+A(1)H'''') \\ &+ n^2x^2(11A(1)+18A'(1) \\ &+ 8A(1)H''(1)+6A''(1)+9A'(1)H''(1) \\ &+ 3A(1)(H''(1))^2+4A(1)H'''(1))+n^3x^3(6A(1)+4A'(1) \\ &+ 6A(1)H''(1))+n^4x^4A(1)]e^n xH(1).\end{aligned}$$

The proof of Lemma 2.3 is obvious using these relation. □

Lemma 2.4. *The operator (1.3) satisfies the following relation:*

$$\begin{aligned}T_n(\psi_x^4(t);x) &= \left(3+14H''(1)+\frac{3A'(1)H''(1)}{A(1)}+3(H'')^2+4H''(1)\right)\frac{x^2}{n^2} \\ &+ \left(6+\frac{6A''(1)}{A(1)}+\frac{14A'(1)}{A(1)}+\frac{6A''(1)H''(1)}{A(1)}+\frac{18A'(1)H''(1)}{A(1)}\right. \\ &+ \frac{4A''(1)H''(1)}{A(1)}+\frac{6A''(1)H''(1)}{A(1)} \\ &+ \left.6H'''(1)+11H''(1)+H''''(1)\right)\frac{x}{n^3} \\ &+ \frac{6A'(1)+11A''(1)+A''''(1)}{n^4A(1)}.\end{aligned}$$

Proof. Proof of this relation can be obtained using Lemma 2.1 and linearity property of the operators

$$T_n((t-x)^4;x) = T_n(t^4;x) - 4xT_n(t^3;x) + 6x^2T_n(t^2;x) - 4x^3T_n(t;x) + T_n(1;x).$$

□

3. The Voronovskaya type theorem for T_n

Theorem 3.1. *Let $f \in C^2[0,b]$. Then $\forall\ x \in [0,b]$, we have*

$$\lim_{n\to\infty} n\{T_n(f;x) - f(x)\} = \frac{A'(1)}{A(1)} f'(x) + (H''(1)+1)x\frac{f''(x)}{2!}.$$

Proof. Let $x_0 \in [0,b]$ be a fixed point. Then for $f \in C^2[0,b]$ and $t \in [0,b]$ we have by Taylor's formula

$$f(t) = f(x_0) + f'(x_0)(t-x_0) + \frac{1}{2}f''(x_0)(t-x)^2 + \varphi(t;x_0)(t-x_0)^2\ ,$$

where $\varphi(t;x_0) \in C[0,b]$ and $\lim_{t\to x_0} g(t;x_0) = 0$. Now, applying the operator on both the side and in the light of linearity property, we have

$$\begin{aligned} T_n(f;x) &= f(x_0)T_n(1;x_0) + f'(x_0)T_n((t-x_0);x_0) + \frac{1}{2}f''(x_0)T_n((t-x)^2;x_0) \\ &\quad + T_n(\varphi(t;x_0)(t-x_0)^2;x_0)). \end{aligned}$$

Subtract $f(x_0)$ and then on multiplying by n both side, we obtain

$$\begin{aligned} n\{T_n(f;x_0) - f(x_0)\} &= f'(x_0)nT_n((t-x_0);x_0) + \frac{f''(x_0)}{2}nT_n((t-x_0)^2;x_0) \\ &\quad + nT_n\Big(\varphi(t;x_0)(t-x)^2;x_0\Big). \end{aligned}$$

We have

$$\begin{aligned} \lim_{n\to\infty} n\{T_n(f;x) - f(x)\} &= \frac{A'(1)}{A(1)}f'(x) + (H''(1)+1)x\frac{f''(x)}{2!} \\ &\quad + \lim_{n\to\infty} nT_n\Big(\varphi(t;x_0)(t-x)^2;x_0\Big). \end{aligned}$$

Using Holder's inequality. The last term can be given by

$$nT_n\Big(\varphi(t;x_0)(t-x)^2;x_0\Big) \le n^2T_n\Big((t-x)^4;x_0\Big)T_n\Big(\varphi(t;x_0)^2;x_0\Big).$$

Let $\eta(t;x_0)=\varphi^2(t;x_0)$. Then $\lim \eta(t;x_0)=\lim \varphi^2(t;x_0)=0$ as $n\to\infty$. By using

$$\lim_{n\to\infty} n^2T_n(\psi_x^4(t);x)=\left(3+14H''(1)+\frac{3A'(1)H''(1)}{A(1)}+3(H''(1)^2)+4H''(1)\right)x^2,$$

we get

$$\lim_{n\to\infty} nT_n\left(\varphi(t;x_0)(t-x)^2;x_0\right)=0,$$

which proves the Theorem 3.1. □

4. Weighted approximation

Here, we recall some notation from [11] to prove next result. Let $B_{1+x^2}[0,\infty)=\{f(x):|f(x)|\le M_f(1+x^2), 1+x^2$ is weight function, M_f is a constant depending on f and $x\in[0,\infty)\}$, $C_{1+x^2}[0,\infty)$ is the space of continuous function in $B_{1+x^2}[0,\infty)$ with the norm $\|f(x)\|_{1+x^2}=\sup\limits_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}$ and $C^k_{1+x^2}[0,\infty)=\{f\in C_{1+x^2}:\lim\limits_{|x|\to\infty}\frac{f(x)}{1+x^2}=k$, where k is a constant depending on $f\}$.

Modulus of continuity for the function f defined on closed interval $[0,a]$ with $a>0$ is denoted as follows

$$\omega_a(f,\delta)=\sup_{|t-x|\le\delta}\ \sup_{x,t\in[0,a]}|f(t)-f(x)|. \tag{4.1}$$

Theorem 4.1. *Let $f\in C_{1+x^2}[0,\infty)$ and $\omega_{b+1}(f;\delta)$ be its modulus of continuity defined on $[0,b+1]\subset[0,\infty)$. Then, we have*

$$\|T_n(f;x)-f(x)\|_{C[0,b]}\le 6M_f(1+b^2)\delta_n(b)+2\omega_{b+1}(f;\sqrt{\delta_n(b)}),$$

where $\delta_n(b)=T_n(\psi_b^2;b)$.

Proof. From ([12], p. 378), for $x\in[0,b]$ and $t\in[0,\infty)$, we have

$$|f(t)-f(x)|\le 6M_f(1+b^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f;\delta).$$

This implies that

$$|T_n(f;x)-f(x)|\le 6M_f(1+b^2)T_n((t-x)^2;x)+\left(1+\frac{T_n(|t-x|;x)}{\delta}\right)\omega_{b+1}(f;\delta).$$

Thus, using Lemma 2.4, for $x\in[0,b]$, we have

$$|T_n(f;x) - f(x)| \le 6M_f(1+b^2)\delta_n(b) + \left(1 + \frac{\sqrt{\delta_n(b)}}{\delta}\right)\omega_{b+1}(f;\delta).$$

Choosing $\delta = \delta_n(b)$, we arrive at the desired result. □

Theorem 4.2. *If the operators T_n defined by (1.3) from $C^k_{1+x^2}[0,\infty)$ to $B_{1+x^2}[0,\infty)$ satisfying the conditions*

$$\lim_{n\to\infty} \|T_n(e_i;x) - x^i\|_{1+x^2} = 0, \quad i = 0,1,2,$$

then for each $C^k_{1+x^2}[0,\infty)$

$$\lim_{n\to\infty} \|T_n(f;x) - f\|_{1+x^2} = 0.$$

Proof. To prove this Theorem, it is enough to show that

$$\lim_{n\to\infty} \|T_n(e_i;x) - x^i\|_{1+x^2} = 0, \quad i = 0,1,2.$$

From Lemma 2.2, we have

$$\|T_n(e_0;x) - x^0\|_{1+x^2} = \sup_{x\in[0,\infty)} \frac{|T_n(1;x) - 1|}{1+x^2} = 0 \text{ for } i = 0.$$

For $i = 1$

$$\begin{aligned} \|T_n(e_1;x) - x^1\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{\frac{A'(1)}{nA(1)}}{1+x^2} \\ &= \frac{A'(1)}{nA(1)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}. \end{aligned}$$

This implies that $\|T_n(e_1;x) - x^1\|_{1+x^2} \to 0$ an $n \to \infty$. For $i = 2$

$$\begin{aligned} \|T_n(e_2;x) - x^2\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{\left|\left(\frac{2A'(1)}{A(1)} + H''(1) + 1\right)\frac{x}{n} + \frac{A'(1)+A''(1)}{n^2A(1)}\right|}{1+x^2} \\ &\le \frac{\left(\frac{2A'(1)}{A(1)} + H''(1) + 1\right)}{n} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ &+ \frac{A'(1)+A''(1)}{n^2A(1)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}. \end{aligned}$$

Which shows that $\|T_n(e_2;x) - x^2\|_{1+x^2} \to 0$ an $n \to \infty$. □

Let $f \in C_\rho^k[0,\infty)$, Yüksel and Ispir [13] introduced weighted modulus of continuity as follows

$$\Omega(f;\delta) = \sup_{x\in[0,\infty),0<h\leq\delta} \frac{|f(x+h)-f(x)|}{1+(x+h)^2}.$$

Theorem 4.3. *Let $f \in C_{1+x^2}^k[0,\infty)$. Then*

(i) $\Omega(f;\delta)$ *is a monotone increasing function of* δ*;*

(ii) $\lim_{\delta\to 0^+} \Omega(f;\delta) = 0$;

(iii) for each $m \in \mathbb{N}$, $\Omega(f;m\delta) \leq m\Omega(f;\delta)$*;*

(iv) for each $\lambda \in [0,\infty)$, $\Omega(f;\lambda\delta) \leq (1+\lambda)\Omega(f;\delta)$

and for $t,x \in [0,\infty)$, *one get*

$$|f(t)-f(x)| \leq 2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\left(1+x^2\right)\left(1+(t-x)^2\right)\Omega(f;\delta). \tag{4.2}$$

Theorem 4.4. *Let $f \in C_{1+x^2}^k[0,\infty)$. Then, we have*

$$\sup_{x\in[0,\infty)} \frac{|T_n(f;x)-f(x)|}{(1+x^2)^3} \leq C\left(1+\frac{1}{n}\right)\Omega\left(f;\frac{1}{\sqrt{n}}\right),$$

where $C>0$ is a constant.

Proof. Using (4.2) and $x,t \in (0,\infty)$, we have

$$\begin{aligned} |T_n(f;x)-f(x)| &\leq 2\left(1+\tfrac{T_n(|t-x|;x)}{\delta}\right)\left(1+\delta^2\right)\left(1+x^2\right) \\ &\times\left(1+T_n((t-x)^2;x)\right)\Omega(f;\delta). \end{aligned} \tag{4.3}$$

Applying Cauchy-Schwarz inequality for (4.2), we get

$$\begin{aligned} |T_n(f;x)-f(x)| \leq\ & 2\left(1+\delta^2\right)(1+x^2)\Omega(f;\delta)\Bigg(1+T_n((t-x)^2;x) \\ +\ & \tfrac{\sqrt{T_n((t-x)^2;x)}}{\delta} + \tfrac{\sqrt{T_n((t-x)^2;x)T_n((t-x)^4;x)}}{\delta}\Bigg). \end{aligned} \tag{4.4}$$

Using Lemma 2.2 and Lemma 2.4, we get

$$T_n((t-x)^2;x) \leq C_1 \frac{(1+x)}{n} \text{ and } T_n((t-x)^4;x) \leq C_2 \frac{(1+x+x^2+x^3)}{n}. \tag{4.5}$$

From and (4.3), we have

$$\begin{aligned} |T_n(f;x) - f(x)| &\leq 2\left(1+\delta^2\right)(1+x^2)\Omega(f;\delta)\Bigg(1 + C_1\frac{(1+x)}{n} \\ &+ \frac{\sqrt{C_1\frac{(1+x)}{n}}}{\delta} + \frac{\sqrt{C_1\frac{(1+x)}{n}C_2\frac{(1+x+x^2+x^3)}{n}}}{\delta}\Bigg). \end{aligned}$$

On choosing $\delta = \frac{1}{\sqrt{n}}$ and $C = \{1 + C_1 + \sqrt{C_1} + \sqrt{C_1 C_2}\}$, we get the required result. □

Theorem 4.5. *For $f \in C^k_{1+x^2}[0,\infty)$ and $\theta > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|T_n(f;x)| - f(x)}{(1+x^2)^{1+\theta}} = 0.$$

Proof. For any fixed real number $x_0 > 0$, one has say

$$\begin{aligned} \sup_{x\in[0,\infty)} \frac{|T_n(f;x)|-f(x)}{(1+x^2)^{1+\theta}} &\leq \sup_{x\leq x_0} \frac{|T_n(f;x)|-f(x)}{(1+x^2)^{1+\theta}} + \sup_{x\geq x_0} \frac{|T_n(f;x)|-f(x)}{(1+x^2)^{1+\theta}} \\ &\leq \||T_n(f;x)| - f(x)\|_{C[0,x_0]} \\ &+ \|f\|_{1+x^2} \sup_{x\geq x_0} \frac{|T_n(1+t^2;x)|}{(1+x^2)^{1+\theta}} + \sup_{x\geq x_0} \frac{|f(x)|}{(1+x^2)^{1+\theta}} \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4.6}$$

Since $|f(x)| \leq \|f\|_{1+x^2}(1+x^2)$, we have

$$\begin{aligned} I_3 &= \sup_{x\geq x_0} \frac{|f(x)|}{(1+x^2)^{1+\theta}} \\ &\leq \sup_{x\geq x_0} \frac{\|f\|_{1+x^2}(1+x^2)}{(1+x^2)^{1+\theta}} \leq \frac{\|f\|_{1+x^2}}{(1+x^2)^{\theta}}. \end{aligned}$$

Let $\epsilon > 0$ be arbitrary real number. Then, from Theorem 4.2 there exists $n_1 \in \mathbb{N}$ such that

$$\begin{aligned} I_2 &< \frac{1}{(1+x^2)^\theta}\|f\|_{1+x^2}\left(1 + x^2 + \frac{\epsilon}{3\|f\|_{1+x^2}}\right) \text{ for all } n_1 \geq n, \\ &< \frac{\|f\|_{1+x^2}}{(1+x^2)^\theta} + \frac{\epsilon}{3} \text{ for all } n_1 \geq n. \end{aligned}$$

This implies that

$$I_2 + I_3 < 2\frac{\|f\|_{1+x^2}}{(1+x^2)^\theta} + \frac{\epsilon}{3}.$$

Next, let for a large value of x_0, we have $\frac{\|f\|_{1+x^2}}{(1+x^2)^\theta} < \frac{\epsilon}{6}$.

$$I_2 + I_3 < \frac{2\epsilon}{3} \text{ for all } n_1 \geq n. \tag{4.7}$$

From Theorem 4.2, there exists $n_2 > n$ in such a way

$$I_1 = \|T_n(f) - f\|_{C[0,x_0]} < \frac{\epsilon}{3} \text{ for all } n_2 \geq n. \tag{4.8}$$

Let $n_3 = max(n_1, n_2)$. Then, combining (4.6), (4.7) and (4.8), we have

$$\sup_{x\in[0,\infty)} \frac{|T_n(f;x)| - f(x)}{(1+x^2)^{1+\theta}} < \epsilon.$$

Hence, the proof of Theorem 4.5 is completed. □

5. A-statistical approximation

Gadjiev et al. [14] was the first who introduced Statistical approximation theorems in operators theory. Here, we recall same notation from [14], let $A = (a_{nk})$ be a non-negative infinite suitability matrix. For a given sequence $x := (x_k)$, the A-transform of x denoted by $Ax : ((Ax)_n)$ is defined as

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k,$$

provided the series converges for each n. A is said to be regular if $\lim(Ax)_n = L$ whenever $\lim x = L$. Then $x = (x_n)$ is said to be a A-statistically convergent to L i.e. $st_A - \lim\ x = L$ if for every $\epsilon > 0$, $\lim_n \sum_{k:|x_k - L|\geq\epsilon} a_{nk} = 0$.

Theorem 5.1. *Let $A = (a_{nk})$ be a non-negative regular suitability matrix and $x \geq 0$. Then, we have*

$$st_A - \lim_n \|T_n(f;x) - f\|_{1+x^{2+\lambda}} = 0, \text{ for all } f \in C^k_{1+x^{2+\lambda}}[0,\infty) \text{ and } \lambda > 0.$$

Proof. From ([15], p. 191, Th. 3), it is sufficient to show that for $\lambda = 0$

$$st_A - \lim_n \|T_n(e_i;x) - e_i\|_{1+x^2} = 0, \text{ for } i \in \{0,1,2\}. \tag{5.1}$$

Using Lemma 2.2, we have

$$\begin{aligned}\|T_n(e_1;x) - x\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\frac{A'(1)}{nA(1)}\right| \\ &= \frac{A'(1)}{nA(1)} \sup_{x\in[0,\infty)} \frac{1}{1+x^2}.\end{aligned}$$

Now, for a given $\epsilon > 0$, we define the following sets

$$\begin{aligned} M_1 : &= \left\{ n : \|T_n(e_1;x) - x\| \geq \epsilon \right\}, \\ M_2 : &= \left\{ n : \frac{A'(1)}{nA(1)} \geq \epsilon \right\}. \end{aligned}$$

This implies that $M_1 \subseteq M_2$, which shows that $\sum_{k \in M_1} a_{nk} \leq \sum_{k \in M_2} a_{nk}$. Hence, we have

$$st_A - \lim_n \|T_n(e_1;x) - x\|_{1+x^2} = 0. \tag{5.2}$$

For $i = 2$ and using Lemma 2.2, we have

$$\begin{aligned} \|T_n(e_2;x) - x^2\|_{1+x^2} &= \sup_{x \in [0,\infty)} \frac{x}{1+x^2} \left| \left(\frac{2A'(1)}{A(1)} + H''(1) + 1 \right) \frac{1}{n} \right. \\ &+ \frac{A'(1) + A''(1)}{n^2 A(1)} \sup_{x \in [0,\infty)} \frac{1}{1+x^2}. \end{aligned}$$

For a given $\varepsilon > 0$, we have the following sets

$$\begin{aligned} T_1 : &= \left\{ n : \left\| T_n(e_2;x) - x^2 \right\| \geq \epsilon \right\}, \\ T_2 : &= \left\{ n : \left(\frac{2A'(1)}{A(1)} + H''(1) + 1 \right) \frac{1}{n} \geq \frac{\epsilon}{2} \right\}, \\ T_3 : &= \left\{ n : \frac{A'(1) + A''(1)}{n^2 A(1)} \geq \frac{\epsilon}{2} \right\}. \end{aligned}$$

This implies that $T_1 \subseteq T_2 \bigcup T_3$. By which, we get

$$\sum_{k \in T_1} a_{nk} \leq \sum_{k \in T_2} a_{nk} + \sum_{k \in T_3} a_{nk}.$$

As $n \to \infty$, we have

$$st_A - \lim_n \|T_n(e_2;x) - x^2\|_{1+x^2} = 0. \tag{5.3}$$

This completes the proof of Theorem 5.1. □

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Ergodic Properties of Random Infinite Products of Nonexpansive Mappings

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ABSTRACT: In this paper we are concerned with the asymptotic behavior of random (unrestricted) infinite products of nonexpansive self-mappings of closed and convex subsets of a complete hyperbolic space. In contrast with our previous work in this direction, we no longer assume that these subsets are bounded. We first establish two theorems regarding the stability of the random weak ergodic property and then prove a related generic result. These results also extend our recent investigations regarding nonrandom infinite products.

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1. Introduction and preliminaries

It is well known that (unrestricted) infinite products of operators find applications in many areas of mathematics (see, for example, [1, 2, 3, 4, 5, 9, 11] and the references mentioned therein). In this paper we establish weak ergodic theorems concerning the asymptotic behavior of random (that is, unrestricted) infinite products of nonexpansive mappings on closed and convex subsets of a Banach space which are not necessarily bounded. These theorems continue our previous work [9], where we assumed that the mappings under consideration act on a bounded set. They also extend the results of [12], which were obtained for *nonrandom* infinite products.

More precisely, our paper contains three theorems. The first two show that if the random weak ergodic property (see the definition below) holds for a sequence of nonexpansive mappings, then it is stable under small perturbations of these mappings. In our second theorem the perturbed mappings are also nonexpansive, while in the first one the perturbations can be arbitrary. The third theorem establishes the random weak ergodic property for a generic sequence of nonexpansive mappings. Namely, we show that in an appropriate space of sequences of nonexpansive mappings there exists a subset which is a countable intersection of open and everywhere dense sets such that each sequence belonging to this subset has the random weak ergodic property. Such an approach is common in nonlinear analysis [9, 10, 11]. Thus, instead of considering a certain convergence property for a single sequence of operators, we investigate it for a space of all such sequences equipped with some natural metric, and show that this property holds for most of these sequences in the sense of Baire category. This allows us to establish convergence without restrictive assumptions on the space and on the operators themselves.

As a matter of fact, it turns out that our results also hold for nonexpansive self-mappings of closed and convex sets in complete hyperbolic spaces, an important class of metric spaces the definition of which we now recall for the reader's convenience.

Let (X, ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c : R^1 \to X$ is a *metric embedding* of R^1 into X if $\rho(c(s), c(t)) = |s - t|$ for all real s and t. The image of R^1 under a metric embedding is called a *metric line*. The image of a real interval $[a, b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping is called a *metric segment*.

Assume that (X, ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X, there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x, y]$. For each $0 \le t \le 1$, there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \text{ and } \rho(z, y) = (1 - t)\rho(x, y).$$

This point will be denoted by $(1-t)x \oplus ty$. We say that X, or more precisely (X, ρ, M), is a *hyperbolic space* if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \le \frac{1}{2}\rho(y, z)$$

for all x, y and z in X. An equivalent requirement is that

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z) \le \frac{1}{2}(\rho(x, w) + \rho(y, z))$$

for all x, y, z and w in X. This inequality, in its turn, implies that

$$\rho((1 - t)x \oplus ty, (1 - t)w \oplus tz) \le (1 - t)\rho(x, w) + t\rho(y, z)$$

for all points x, y, z and w in X, and all numbers $0 \le t \le 1$.

It is clear that all normed linear spaces are hyperbolic in this sense. A discussion of more examples of hyperbolic spaces and, in particular, of the Hilbert ball with the hyperbolic metric can be found, for example, in [6, 7, 8].

We call a set $K \subset X$ *ρ-convex* if $[x, y] \subset K$ for all x and y in K.

Suppose that (X, ρ, M) is a complete hyperbolic space and that K is a nonempty, closed and ρ-convex subset of the space X. Denote by $\mathcal{A}$ the collection of all operators $T : K \to K$ which satisfy

$$\rho(T(x), T(y)) \le \rho(x, y) \text{ for all } x, y \in K. \tag{1.1}$$

Denote by $\mathcal{M}$ the set of all sequences of operators $\{T_i\}_{i=1}^{\infty} \subset \mathcal{A}$. For every sequence of operators $\{B_i\}_{i=1}^{\infty} \in \mathcal{M}$ and every pair of integers $p > n \geq 1$, we define compositions of the corresponding operators by

$$\prod_{i=n}^{p} B_i := B_p \cdots B_n.$$

For every point $x \in K$ and every positive number r, set

$$B(x, r) := \{y \in K : \rho(x, y) \leq r\}.$$

Fix a point $\theta \in K$. We equip the set $\mathcal{M}$ with the uniformity determined by the following base:

$$\mathcal{U}(n) := \{(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) \in \mathcal{M} \times \mathcal{M} :$$
$$\rho(A_t(x), B_t(x)) \leq n^{-1} \text{ for all } x \in B(\theta, n) \text{ and all integers } t \geq 1\}, \tag{1.2}$$

where $n \geq 1$ is an integer. It is not difficult to see that the uniform space $\mathcal{M}$ is metrizable (by a metric d) and complete. In principle, one can obtain an explicit expression for this metric d, but we do not need it because in our case it is more convenient to use the uniformity itself.

Denote by $I : K \to K$ the identity operator; that is, $I(x) = x$ for all $x \in K$.

In this paper we are interested in those sequences of mappings in $\mathcal{M}$ which are uniformly bounded on bounded sets.

Proposition 1.1. *Let $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$, $x \in K$ and assume that $\{A_t(x)\}_{t=1}^{\infty}$ is a bounded sequence. Then for every $y \in K$, the sequence $\{A_t(y)\}_{t=1}^{\infty}$ is bounded and*

$$\sup\{\rho(\theta, A_t(y)) : t = 1, 2, \dots\} \leq \sup\{\rho(\theta, A_t(x)) : t = 1, 2, \dots\} + \rho(x, y).$$

Proof. Clearly, the real sequence $\{\rho(\theta, A_t(x))\}_{t=1}^{\infty}$ is bounded. Let $y \in K$. Then in view of (1.1), for every integer $t \geq 1$,

$$\rho(\theta, A_t(y)) \leq \rho(\theta, A_t(x)) + \rho(A_t(x), A_t(y)) \leq \rho(\theta, A_t(x)) + \rho(x, y).$$

Proposition 1.1 now follows immediately. □

We denote by $\mathcal{M}_n$ the set of all sequences $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ such that the sequence $\{A_t(\theta) : t = 1, 2, \dots\}$ is bounded. Clearly, $\mathcal{M}_n$ is a closed and open subset of the complete metric space $(\mathcal{M}, d)$. In this paper we focus on the complete metric space $(\mathcal{M}_n, d)$.

For every point $z \in K$ and every nonempty set $D \subset K$, set

$$\rho(z, D) := \inf\{\rho(z, \xi) : \xi \in D\}.$$

We say that a sequence of mappings $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ possesses the *weak ergodic property* [12] (WEP, for short) if for every pair of positive numbers ϵ, s, there exists an integer $n_0 \geq 1$ such that for every pair of points $x, y \in B(\theta, s)$, we have

$$\rho(A_{n_0} \cdots A_1(x), A_{n_0} \cdots A_1(y)) \leq \epsilon.$$

In [12] we consider the space $\mathcal{M}$ equipped with the uniformity determined by the base

$$\tilde{\mathcal{U}}(n) := \{(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) \in \mathcal{M} \times \mathcal{M} :$$

$$\rho(A_t(x), B_t(x)) \le n^{-1} \text{ for all } x \in B(\theta, n) \text{ and all } t = 1, \dots, n\},$$

where n is a natural number, and show that most sequences in $\mathcal{M}$, in the sense of Baire category, have the WEP and that the WEP is stable under small perturbations.

We say that a sequence of mappings $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ possesses the *random weak ergodic property* (RWEP, for short) if for every pair of positive numbers ϵ, s, there exists an integer $n_0 \ge 1$ such that for every pair of points $x, y \in B(\theta, s)$ and every mapping $r : \{1, \dots, n_0\} \to \{1, 2, \dots\}$, we have

$$\rho(A_{r(n_0)} \cdots A_{r(1)}(x), A_{r(n_0)} \cdots A_{r(1)}(y)) \le \epsilon.$$

In [13] we continue to study the space $\mathcal{M}$ with the uniformity introduced in [12] and show that most sequences in $\mathcal{M}$, in the sense of Baire category, do not have the RWEP and that they display, in fact, *chaotic* asymptotic behavior. In the present paper we show, on the other hand, that the RWEP *does hold generically* in the complete metric space $(\mathcal{M}_n, d)$. We begin with the following stability result.

2. First stability result

Theorem 2.1. *Assume that a sequence of mappings $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ possesses the RWEP and let ϵ, s be positive numbers. Then there exists an integer $n_0 \ge 1$ such that for every natural number $n \ge n_0$, there exists a number $\delta > 0$ such that for every mapping $r : \{1, 2, \dots, n\} \to \{1, 2, \dots\}$ and every pair of sequences $\{x_i\}_{i=0}^{n}, \{y_i\}_{i=0}^{n} \subset K$ satisfying*

$$x_0,\ y_0 \in B(\theta, s) \tag{2.1}$$

and

$$\rho(x_{i+1}, A_{r(i+1)}(x_i)) \le \delta,\ \rho(y_{i+1}, A_{r(i+1)}(y_i)) \le \delta \tag{2.2}$$

for all integers $i = 0, \dots, n-1$, the inequality $\rho(x_i, y_i) \le \epsilon$ is valid for all integers $i = n_0, \dots, n$.

Proof. By definition, there exists an integer $n_0 \ge 1$ such that the following property holds:

(i) for every mappings $r : \{1, \dots, n_0\} \to \{1, 2, \dots\}$ and every pair of points

$$x,\ y \in B(\theta, s),$$

we have

$$\rho(A_{r(n_0)} \cdots A_{r(1)}(x), A_{r(n_0)} \cdots A_{r(1)}(y)) \le \epsilon/2.$$

Let $n \ge n_0$ be a natural number and let

$$\delta = \epsilon(4n)^{-1}. \tag{2.3}$$

Assume that

$$r : \{1, \dots, n\} \to \{1, 2, \dots\}$$

and that two sequences

$$\{x_i\}_{i=0}^n, \{y_i\}_{t=0}^n \subset K$$

satisfy inclusion (2.1) and inequalities (2.2). It then follows from property (i) and (2.1) that

$$\rho(A_{r(n_0)} \cdots A_{r(1)}(x_0), A_{r(n_0)} \cdots A_{r(1)}(y_0)) \le \epsilon/2.$$

When combined with (1.1), this inequality implies that

$$\rho(A_{r(p)} \cdots A_{r(1)}(x_0), A_{r(p)} \cdots A_{r(1)}(y_0)) \le \epsilon/2 \text{ for all } p = n_0, \dots, n. \tag{2.4}$$

Let

$$\{z_i\}_{i=0}^n \in \{\{x_i\}_{i=0}^n,\ \{y_i\}_{i=0}^n\}, \tag{2.5}$$

$$\xi_0 = z_0 \tag{2.6}$$

and let for all $i = 0, \dots, n-1$,

$$\xi_{i+1} = A_{r(i+1)}(\xi_i). \tag{2.7}$$

We claim that for all integers $i = 0, \dots, n$, we have

$$\rho(z_i, \xi_i) \le i\delta. \tag{2.8}$$

We first note that it follows from (2.6) that inequality (2.8) is valid for $i = 0$.

Assume now that $i < n$ is a nonnegative integer and that inequality (2.8) is true. By (1.1), (2.2), (2.5), (2.7) and (2.8), we have

$$\rho(z_{i+1}, \xi_{i+1}) = \rho(z_{i+1}, A_{r(i+1)}(\xi_i))$$

$$\le \rho(z_{i+1}, A_{r(i+1)}(z_i)) + \rho(A_{r(i+1)}(z_i), A_{r(i+1)}(\xi_i))$$

$$\le \delta + \rho(z_i, \xi_i) \le (i+1)\delta.$$

Thus we have shown by induction that inequality (2.8) is indeed true for all integers $i = 0, \dots, n$, as claimed. When combined with (2.5)–(2.7), this implies that for all integers $i = 1, \dots, n$, we have

$$\rho(x_i, A_{r(i)} \cdots A_{r(1)}(x_0)) \le i\delta,$$

$$\rho(y_i, A_{r(i)} \cdots A_{r(1)}(y_0)) \le i\delta$$

and

$$\rho(x_i, y_i) \le \rho(A_{r(i)} \cdots A_{r(1)}(x_0), A_{r(i)} \cdots A_{r(1)}(y_0)) + 2i\delta. \tag{2.9}$$

It now follows from (1.1), (2.3), (2.4) and (2.9) that for all integers $i = n_0, \dots, n$,

$$\rho(x_i, y_i) \le \rho(A_{r(i)} \cdots A_{r(1)}(x_0), A_{r(i)} \cdots A_{r(1)}(y_0)) + 2n\delta \le \epsilon/2 + \epsilon/2.$$

This completes the proof of Theorem 2.1. □

3. Second stability result

Theorem 3.1. *Assume that a sequence of mappings* $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}_n$ *possesses the RWEP and let* ϵ, s *be positive numbers. Then there exists an integer* $n_0 \geq 1$ *and a neighborhood* $\mathcal{U}$ *of* $\{A_t\}_{t=1}^{\infty}$ *in* $\mathcal{M}$ *such that for every sequence of mappings* $\{B_t\}_{t=1}^{\infty} \in \mathcal{U}$*, every mapping* $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$*, every pair of points* $x, y \in B(\theta, s)$ *and every natural number* $n \geq n_0$*, the inequality*

$$\rho(B_{r(n)} \cdots B_{r(1)}(x), B_{r(n)} \cdots B_{r(1)}(y)) \leq \epsilon$$

is true.

Proof. Theorem 2.1 implies that there exist an integer $n_0 \geq 1$ and a real number $\delta \in (0, 1)$ such that the following property holds:

(i) for every mapping $r : \{1, , \dots, n_0\} \to \{1, 2, \dots\}$ and every pair of sequences $\{x_i\}_{t=0}^{n_0}, \{y_i\}_{t=0}^{n_0} \subset K$ satisfying

$$x_0,\ y_0 \in B(\theta, s)$$

and

$$\rho(x_{i+1}, A_{r(i+1)}(x_i)) \leq \delta,\ \rho(y_{i+1}, A_{r(i+1)}(y_i)) \leq \delta$$

for all $i = 0, \dots, n_0 - 1$, we have $\rho(x_{n_0}, y_{n_0}) \leq \epsilon$.

Define

$$F_0 := B(\theta, s) \tag{3.1}$$

and for all $i = 1, \dots, n_0 - 1$, define

$$F_{i+1} := \{y \in K :\ \rho(y, \cup\{A_p(F_i) :\ p = 1, 2, \dots\}) \leq 1\}. \tag{3.2}$$

We now show by induction that all the sets $F_0, \dots, F_{n_0}$ are bounded. In view of (3.1), F_0 is clearly bounded. Assume that $i < n_0$ is a nonnegative integer and that the set F_i is bounded. Then there exists $M_0 > 0$ such that

$$\rho(\theta, z) \leq M_0 \text{ for all } z \in F_i. \tag{3.3}$$

By (3.3) and Proposition 1.1, for each $y \in F_i$,

$$\sup\{\rho(\theta, A_p(y)) :\ p = 1, 2, \dots\}$$
$$\leq \sup\{\rho(\theta, A_p(\theta)) :\ p = 1, 2, \dots\} + \rho(\theta, y)$$
$$\leq \sup\{\rho(\theta, A_p(\theta)) :\ p = 1, 2, \dots\} + M_0.$$

This implies that

$$\cup\{A_p(F_i) :\ p = 1, 2, \dots\} \subset B(\theta, M_0 + \sup\{\rho(\theta, A_p(\theta)) :\ p = 1, 2, \dots\})$$

and that the set F_{i+1} is bounded. Thus we have shown by induction that all the sets $F_0, \dots, F_{n_0}$ are bounded, as asserted.

Next, choose $s_0 > s$ such that

$$F_i \subset B(\theta, s_0),\ i = 0, \dots, n_0. \tag{3.4}$$

There exists a neighborhood $\mathcal{U}$ of $\{A_t\}_{t=1}^{\infty}$ in $\mathcal{M}$ such that the following property holds:

(ii) for every sequence of mappings $\{B_t\}_{t=1}^{\infty} \in \mathcal{U}$, every point $x \in B(\theta, s_0)$ and every integer $t \geq 1$, we have

$$\rho(B_t(x), A_t(x)) \leq \delta.$$

Assume that

$$\{B_t\}_{t=1}^{\infty} \in \mathcal{U},\ r : \{1, 2, \dots\} \to \{1, 2, \dots\} \ \text{ and } \ x \in B(\theta, s). \tag{3.5}$$

We claim that for all integers $i = 1, \dots, n_0$,

$$\prod_{j=1}^{i} B_{r(j)}(x) \in F_i. \tag{3.6}$$

Property (ii), (3.1), (3.2) and (3.5) imply that (3.6) is indeed true for $i = 1$.

Assume now that $i < n_0$ is a nonnegative integer and that (3.6) holds. Then it follows from property (ii), (3.2), (3.4), (3.5) and (3.6) that

$$\rho(B_{r(i+1)} \prod_{j=1}^{i} B_{r(j)}(x), A_{r(i+1)} \prod_{j=1}^{i} B_{r(j)}(x)) \leq \delta,$$

$$\rho(B_{r(i+1)} \prod_{j=1}^{i} B_{r(j)}(x), A_{r(i+1)}(F_i)) \leq \delta \leq 1$$

and that

$$\prod_{j=1}^{i+1} B_{r(j)}(x) \in F_{i+1}. \tag{3.7}$$

Thus we have shown by induction that for all integers $i = 1, \dots, n_0$, inclusion (3.6) indeed holds. When combined with (3.4), this implies that

$$\prod_{j=1}^{i} B_{r(j)}(x) \in B(\theta, s_0),\ i = 1, \dots, n_0. \tag{3.8}$$

It follows from property (ii) and inclusion (3.8) that for all integers $i = 0, \dots, n_0 - 1$ and every point $y \in B(\theta, s)$, we have

$$\rho(B_{r(i+1)} \prod_{j=1}^{i} B_{r(j)}(y), A_{r(i+1)} \prod_{j=1}^{i} B_{r(j)}(y)) \leq \delta$$

(here we assume that $\prod_{j=1}^{i} B_{r(j)} = I$ if $i = 0$). Thus we have shown that if (3.5) holds and $y \in B(\theta, s)$, then the sequences $\{x_i\}_{i=0}^{n_0}$ and $\{y_i\}_{i=0}^{n_0}$ defined by

$$x_0 = x,\ y_0 = y,\ x_i = \prod_{j=1}^{i} B_{r(j)}(x),\ y_i = \prod_{j=1}^{i} B_{r(j)}(y),\ i = 1, \dots, n_0,$$

satisfy the conditions assumed in property (i). This leads to the inequality

$$\epsilon \ge \rho(x_{n_0}, y_{n_0}) = \rho(\prod_{j=1}^{n_0} B_{r(j)}(x), \prod_{j=1}^{n_0} B_{r(j)}(y)),$$

which in its turn implies that

$$\epsilon \ge \rho(\prod_{j=1}^{n} B_{r(j)}(x), \prod_{j=1}^{n} B_{r(j)}(y))$$

for each integer $n \ge n_0$. This completes the proof of Theorem 3.1. □

4. Generic result

Theorem 4.1. *There exists a set $\mathcal{F} \subset \mathcal{M}_n$, which is a countable intersection of open and everywhere dense subsets of $\mathcal{M}_n$, such that every sequence of mappings $\{A_t\}_{t=1}^{\infty} \in \mathcal{F}$ possesses the RWEP.*

Proof. Let $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}_n$ and $\gamma \in (0,1)$. For every natural number t, define

$$A_t^{(\gamma)}(x) := (1-\gamma)A_t(x) \oplus \gamma\theta,\ x \in K. \tag{4.1}$$

In view of (1.1) and (4.1), for every natural number t and every pair of points $x, y \in K$, we have

$$\begin{aligned} &\rho(A_t^{(\gamma)}(x), A_t^{(\gamma)}(y)) \\ &= \rho((1-\gamma)A_t(x) \oplus \gamma\theta, (1-\gamma)A_t(y) \oplus \gamma\theta) \\ &\le (1-\gamma)\rho(A_t(x), A_t(y)) \le (1-\gamma)\rho(x,y). \end{aligned} \tag{4.2}$$

This implies that $\{A_t^{(\gamma)}\}_{t=1}^{\infty} \in \mathcal{M}$. By (4.1), for every natural number $t \ge 1$, we have

$$\rho(A_t^{(\gamma)}(\theta), \theta) = \rho((1-\gamma)A_t(\theta) \oplus \gamma\theta, \theta) \le (1-\gamma)\rho(A_t(\theta), \theta).$$

This implies that $\{A_t^{(\gamma)}\}_{t=1}^{\infty} \in \mathcal{M}_n$.

We claim that $\{A_t^{(\gamma)}\}_{t=1}^{\infty}$ has the RWEP. To this end, let ϵ, s be positive numbers. Choose an integer $n_0 \ge 1$ for which

$$2s(1-\gamma)^{n_0} < \epsilon. \tag{4.3}$$

Assume that $r : \{1, \dots, n_0\} \to \{1, 2, \dots\}$ and that $x, y \in B(\theta, s)$. It follows from (4.2) and (4.3) that

$$\rho(A^{(\gamma)}_{r(n_0)} \cdots A^{(\gamma)}_{r(1)}(x), A^{(\gamma)}_{r(n_0)} \cdots A^{(\gamma)}_{r(1)}(y))$$
$$\leq (1-\gamma)^{n_0}\rho(x, y) \leq 2s(1-\gamma)^{n_0} < \epsilon.$$

Therefore $\{A^{(\gamma)}_t\}_{t=1}^{\infty}$ indeed possesses the RWEP, as claimed.

By (4.1), for every natural number t and every point $x \in K$, we have

$$\rho(A^{(\gamma)}_t(x), A_t(x)) = \rho((1-\gamma)A_t(x) \oplus \gamma\theta, A_t(x)) \leq \gamma\rho(A_t(x), \theta). \tag{4.4}$$

We claim that

$$\{A^{(\gamma)}_t\}_{t=1}^{\infty} \to \{A_t\}_{t=1}^{\infty} \text{ in } \mathcal{M}_n \text{ as } \gamma \to 0^+.$$

Let ϵ be a positive number and let $m \geq 1$ be an integer. Choose a number $\gamma_0 \in (0, 1)$ satisfying

$$\gamma_0(m + \sup\{\rho(A_t(\theta), \theta) : \ t = 1, 2, \dots\}) < \epsilon. \tag{4.5}$$

Assume that $\gamma \in (0, \gamma_0)$. In view of (4.1), (4.4) and (4.5), for every point $x \in B(\theta, m)$ and each natural number t, we have

$$\rho(A^{(\gamma)}_t(x), A_t(x)) \leq \gamma\rho(A_t(x), \theta)$$
$$\leq \gamma_0(\rho(A_t(x), A_t(\theta)) + \rho(A_t(\theta), \theta))$$
$$\leq \gamma_0((\rho(x, \theta) + \rho(A_t(\theta), \theta)) \leq \gamma_0(m + \rho(A_t(\theta), \theta)) < \epsilon.$$

Thus the set

$$\{\{A^{(\gamma)}_t\}_{t=1}^{\infty} : \ \{A_t\}_{t=1}^{\infty} \in \mathcal{M}_n, \ \gamma \in (0, 1)\}$$

is everywhere dense in $\mathcal{M}_n$ and its elements possess the RWEP.

Let $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}_n$, $\gamma \in (0, 1)$ and let $q \geq 1$ be an integer. Theorem 3.1 implies that there exist an integer $n(\{A_t\}_{t=1}^{\infty}, \gamma, q) \geq 1$ and an open neighborhood $\mathcal{U}(\{A_t\}_{t=1}^{\infty}, \gamma, q)$ of $\{A^{(\gamma)}_t\}_{t=1}^{\infty}$ in the metric space $\mathcal{M}_n$ such that the following property holds:

(i) for every sequence of mappings $\{B_t\}_{t=1}^{\infty} \in \mathcal{U}(\{A_t\}_{t=1}^{\infty}, \gamma, q)$, every mapping $r : \{1, 2, \dots\} \to \{1, 2, \dots\}$, every pair of points $x, y \in B(\theta, q)$ and every natural number $n \geq n(\{A_t\}_{t=1}^{\infty}, \gamma, q)$, we have

$$\rho(B_{r(n)} \cdots B_{r(1)}(x), B_{r(n)} \cdots B_{r(1)}(y)) \leq q^{-1}.$$

Now define

$$\begin{aligned} \mathcal{F} : \quad = \quad & \cap_{p=1}^{\infty} \cup \{\mathcal{U}(\{A_t\}_{t=1}^{\infty}, \gamma, q) : \ \{A_t\}_{t=1}^{\infty} \in \mathcal{M}_n, \\ & \gamma \in (0, 1), q \geq p \text{ is an integer}\}. \end{aligned} \tag{4.6}$$

It is clear that $\mathcal{F}$ is a countable intersection of open and everywhere dense subsets of $\mathcal{M}_n$.

Let

$$\{B_t\}_{t=1}^{\infty} \in \mathcal{F}, \ s > 0 \ \text{ and } \ \epsilon > 0. \tag{4.7}$$

Choose a natural number p such that

$$p > s + \epsilon^{-1}. \tag{4.8}$$

In view of (4.6) and (4.7), there exist

$$\{A_t\}_{t=1}^{\infty} \in \mathcal{M}_n,\ \gamma \in (0,1), \ \text{ and } \ q \in \{p, p+1, \dots\}$$

such that

$$\{B_t\}_{t=1}^{\infty} \in \mathcal{U}(\{A_t\}_{t=1}^{\infty}, \gamma, q). \tag{4.9}$$

It follows from property (i), (4.8) and (4.9) that for every mapping

$$r : \{1, 2, \dots, \} \to \{1, 2, \dots\},$$

every pair of points

$$x, y \in B(\theta, s) \subset B(\theta, q)$$

and every integer

$$n \geq n(\{A_t\}_{t=1}^{\infty}, \gamma, q),$$

$$\rho(\prod_{i=1}^{n} B_{r(i)}(x), \prod_{i=1}^{n} B_{r(i)}(y)) \leq q^{-1} \leq p^{-1} < \epsilon.$$

Thus $\{B_t\}_{t=1}^{\infty}$ has the RWEP. This completes the proof of Theorem 4.1.

□

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Fekete-Szegő Problems for Certain Class of Analytic Functions Associated with Quasi-Subordination

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ABSTRACT: In this paper, we determine the coefficient estimates and the Fekete-Szegő inequalities for $\mathcal{M}_q^{\alpha}(\gamma, \lambda, \phi)$, the class of analytic and univalent functions associated with quasi-subordination.

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Keywords and Phrases: Univalent functions; Starlike; Convex functions; Subordination and quasi-subordination.

1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of analytic functions defined on the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalized conditions $f(0) = 0 = f'(0)-1$. Let $\mathcal{S}$ be the class of all functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$. So $f(z) \in \mathcal{S}$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}. \tag{1.1}$$

Definition 1.1. For two analytic functions f and g, the function $f(z)$ is subordinate to $g(z)$, written as $f \prec g$, if there exists a Schwarz' function $w(z)$, analytic in $\mathbb{U}$, with $w(0) = 0$, $|w(z)| < 1$, $z \in \mathbb{U}$, such that

$$f(z) = g(w(z)), \quad z \in \mathbb{U}. \tag{1.2}$$

In particular, if the function g is univalent in $\mathbb{U}$, then $f \prec g$ if

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\phi(z)$ be an analytic and univalent function in $\mathbb{U}$ with $\mathrm{Re}f(z) > 0$, $\phi(0) = 1$ and $\phi'(0) > 0$, which maps the unit disk $\mathbb{U}$ on to a region starlike with respect to 1 and symmetric with respect to real axis. So $\phi(z)$ has the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \tag{1.3}$$

where all coefficients are real and $B_1 > 0$. Let $h(z)$ be an analytic function in $\mathbb{U}$ and $|h(z)| \le 1$, such that

$$h(z) = c_0 + c_1 z + c_2 z^2 + \cdots . \tag{1.4}$$

In 1970, Robertson [19] introduced the concept of quasi-subordination as follows:

Definition 1.2. The function f is said to be quasi-subordinate to g, written as

$$f(z) \prec_q g(z), \tag{1.5}$$

if there exist analytic functions h and w, with $|h(z)| \le 1$, $w(0) = 0$ and $|w| < 1$, such that $\frac{f(z)}{h(z)}$ is analytic in $\mathbb{U}$ and

$$\frac{f(z)}{h(z)} \prec g(z), \quad z \in \mathbb{U}. \tag{1.6}$$

Also the above expression is equivalent to

$$f(z) = h(z)g(w(z)), \quad z \in \mathbb{U}. \tag{1.7}$$

Observe that if $h(z) \equiv 1$, then $f(z) = g(w(z))$, so $f(z) \prec g(z)$ in $\mathbb{U}$. Also if $w(z) = z$, then $f(z) = h(z)g(z)$ and it is said to f is majorized by g and written as $f(z) \ll g(z)$ in $\mathbb{U}$. Hence it is obvious that quasi-subordination is a generalization of subordination and majorization (see [19]).

In [15], Ma and Minda gave unified representation of various subclasses of starlike and convex functions by using subordination. They introduced the classes $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ of analytic functions $f \in \mathcal{A}$, that satisfy the conditions $\frac{zf'(z)}{f(z)} \prec \phi(z)$ and $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$ respectively, which includes several well-known subclasses. In particular, if $\phi(z) = \frac{1+Az}{1+Bz}$, $(-1 \le B < A \le 1)$, the class $\mathcal{S}^*(\phi)$ reduces to the class $\mathcal{S}^*[A, B]$, introduced by Janowski [10]. Also for the choice of $\phi(z) = \frac{1+(1-2\alpha)z}{1-z}$ where $(0 \le \alpha < 1)$, the class $\mathcal{S}^*(\phi)$ becomes the class of starlike functions of order α.

Motivated by Ma and Minda, Mohd and Darus [14], introduced two classes $\mathcal{S}_q^*(\phi)$ and $\mathcal{C}_q(\phi)$ of analytic functions $f(z) \in \mathcal{A}$, that satisfying the conditions $\frac{zf'(z)}{f(z)} - 1 \prec_q \phi(z) - 1$ and $\frac{zf''(z)}{f'(z)} \prec_q \phi(z) - 1$ respectively, which are analogous to $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$. They also introduced $\mathcal{M}_q(\alpha, \phi)$ be the class of functions $f(z) \in \mathcal{A}$, that satisfying the condition $(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 \prec_q \phi(z) - 1$, where $0 \le \alpha \le 1$ [14]. This class is analogous of the well-known class of α-convex functions [16].

Recently, El-Ashwah and Kanas [6], introduced and studied the following subclasses by using quasi-subordination:

$$\mathcal{S}_q^*(\gamma, \phi) = \left\{ f \in \mathcal{A} : \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec_q \phi(z) - 1;\ z \in \mathbb{U},\ 0 \ne \gamma \in \mathbb{C} \right\},$$

and

$$\mathcal{C}_q(\gamma, \phi) = \left\{ f \in \mathcal{A} : \frac{1}{\gamma}\frac{zf''(z)}{f'(z)} \prec_q \phi(z) - 1;\ z \in \mathbb{U},\ 0 \ne \gamma \in \mathbb{C} \right\}.$$

For $h(z) = 1$, the classes $\mathcal{S}_q^*(\gamma, \phi) = \mathcal{S}^*(\gamma, \phi)$ and $\mathcal{C}_q(\gamma, \phi) = \mathcal{C}(\gamma, \phi)$, were introduced and studied in [18]. For $\gamma = 1$, the classes $\mathcal{S}_q^*(\gamma, \phi)$ and $\mathcal{C}_q(\gamma, \phi)$, reduce to $\mathcal{S}_q^*(\phi)$ and $\mathcal{C}_q(\phi)$, respectively studied in [14].

Motivated by El-Ashwah and Kanas, we introduce the following subclass of $\mathcal{A}$:

Definition 1.3. For $0 \neq \gamma \in \mathbb{C}$, $\alpha \geq 0$ and $0 \leq \lambda \leq 1$, the class $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$ is defined by

$$\mathcal{M}_q^\alpha(\gamma, \lambda, \phi) = \left\{ f \in \mathcal{A} : \frac{1}{\gamma}\left[(1-\alpha)\frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)} + \alpha\left(1 + \frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)}\right) - 1\right] \prec_q \phi(z) - 1,\ z \in \mathbb{U} \right\}, \tag{1.8}$$

where

$$\mathcal{F}_\lambda(z) = (1-\lambda)f(z) + \lambda z f'(z) = z + \sum_{n=2}^{\infty}\{1 + (n-1)\lambda\}a_n z^n. \tag{1.9}$$

For special choices of α, λ, γ and ϕ, the class $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$ unifies the following known classes.

(i) For $0 \neq \gamma \in \mathbb{C}$, $\lambda = 0$ and $\alpha = 0$, the class $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$ reduces to $\mathcal{S}_q^*(\gamma, \phi)$ studied in [6].

(ii) For $0 \leq \alpha \leq 1$, $\gamma = 1$ and $\lambda = 0$, $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$ reduce to $\mathcal{M}_q^*(\alpha, \phi)$ which was introduced and studied by Mohd and Darus in [14]. In particular, $\alpha = 0$ and $\alpha = 1$ the class $\mathcal{M}_q^*(\alpha, \phi)$ reduce to $\mathcal{S}_q^*(\phi)$ and $\mathcal{C}_q(\phi)$ respectively, which were also studied in [14].

(iii) For $0 \leq \alpha \leq 1$, $\gamma = 1$, $\lambda = 0$ and $h(z) \equiv 1$, the class $\mathcal{M}_q^*(\alpha, \phi)$ reduces to the well-known class of α-convex functions [16].

In 1933, Fekete and Szegő proved that, for $f \in \mathcal{S}$ given by (1.1)

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & if \quad \mu \leq 0, \\ 1 + 2e^{\frac{-2}{1-\mu}}, & if \quad 0 \leq \mu < 1, \\ 4 - 3\mu, & if \quad \mu \geq 1, \end{cases} \tag{1.10}$$

and the result is sharp. The problem of finding the sharp bounds for the non-linear functional $|a_3 - \mu a_2^2|$ of many compact family of functions is popularly known as the Fekete-Szegő problem. Several known authors at different times obtained the sharp bound of the Fekete-Szegő functional $|a_3 - \mu a_2^2|$ for various subclasses of $\mathcal{S}$ (see [5, 6, 7, 22, 23]). In this paper, we determine the coefficient estimates and the Fekete-Szegő inequality of the functions in the class $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$.

Let Ω be the class of the functions of the form:

$$w(z) = w_1 z + w_2 z^2 + \cdots, \tag{1.11}$$

is analytic in the unit disk $\mathbb{U}$ and satisfy the condition $|w(z)| < 1$.

We need the following lemma to prove our main result.

Lemma 1.1. ([*11*], p.10) *If $w \in \Omega$, then for any complex number μ*

$$|w_1| \leq 1, \quad |w_2 - \mu w_1^2| \leq 1 + (|\mu| - 1)|w_1|^2 \leq \max\{1, |\mu|\}.$$

The result is sharp for the functions $w(z) = z$ when $|\mu| \geq 1$ and for $w(z) = z^2$ when $|\mu| < 1$.

2. Main result

Throughout this paper, we assume that the functions $\phi(z), h(z)$ and $w(z)$ defined by (1.3), (1.4) and (1.11), respectively.

Theorem 2.1. *Let $0 \neq \gamma \in \mathbb{C}$, $\alpha \geq 0$ and $0 \leq \lambda \leq 1$. If $f \in \mathcal{A}$ of the form* (1.1) *belongs to the class $\mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$, then*

$$|a_2| \leq \frac{|\gamma| B_1}{(1+\alpha)(1+\lambda)}, \tag{2.12}$$

$$|a_3| \leq \frac{|\gamma| B_1}{2(1+2\alpha)(1+2\lambda)} \left[1 + \max\left\{1, \left(\frac{(1+3\alpha)|\gamma|}{(1+\alpha)^2} B_1 + \frac{|B_2|}{B_1}\right)\right\}\right], \tag{2.13}$$

and for any complex number μ,

$$|a_3 - \mu a_2^2| \leq \frac{|\gamma| B_1}{2(1+2\alpha)(1+2\lambda)} \left[1 + \max\left\{1, \left(|Q| B_1 + \frac{|B_2|}{B_1}\right)\right\}\right], \tag{2.14}$$

where

$$Q = \frac{2\mu(1+2\alpha)(1+2\lambda) - (1+\lambda)^2(1+3\alpha)}{(1+\alpha)^2(1+\lambda)^2}. \tag{2.15}$$

The result is sharp.

Proof. Let $f \in \mathcal{M}_q^\alpha(\gamma, \lambda, \phi)$. Then by Definition 1.3,

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)} + \alpha\left(1 + \frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)}\right) - 1\right] = h(z)(\phi(w(z)) - 1), \tag{2.16}$$

where $F_\lambda(z)$ defined by (1.9).

Using the series expansion of $\mathcal{F}_\lambda(z)$, $\mathcal{F}_\lambda'(z)$ and $\mathcal{F}_\lambda''(z)$ from (1.9), we get

$$\begin{aligned} \tfrac{1}{\gamma}\left[(1-\alpha)\tfrac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)} + \alpha\left(1 + \tfrac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)}\right) - 1\right] &= \tfrac{1}{\gamma}\left[(1+\alpha)(1+\lambda)a_2 z\right. \\ &\left. + \{2(1+2\alpha)(1+2\lambda)a_3 - (1+3\alpha)(1+\lambda)^2 a_2^2\} z^2 + \cdots\right]. \end{aligned} \tag{2.17}$$

Also

$$\phi(w(z)) - 1 = B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + \cdots, \tag{2.18}$$

and

$$h(z)(\phi(w(z)) - 1) = B_1\, c_0\, w_1 z + [B_1 c_1 w_1 + c_0(B_1 w_2 + B_2 w_1^2)]z^2 + \cdots. \quad (2.19)$$

Making use of (2.17), (2.18) and (2.19) in (2.16), and equating the coefficients of z and z^2 in the resulting equation, we get

$$a_2 = \frac{\gamma B_1 c_0}{(1+\alpha)(1+\lambda)}, \quad (2.20)$$

and

$$a_3 = \tfrac{\gamma}{2(1+2\alpha)(1+2\lambda)}\left[(B_1 c_1 w_1 + B_1 c_0 w_2) + c_0\left(B_2 + \tfrac{(1+3\alpha)\gamma}{(1+\alpha)^2} B_1^2 c_0\right) w_1^2\right]. \quad (2.21)$$

Thus, for any complex number μ, we have

$$a_3 - \mu a_2^2 = \frac{\gamma B_1}{2(1+2\alpha)(1+2\lambda)}\left[c_1 w_1 + c_0\left(w_2 + \frac{B_2}{B_1} w_1^2\right) - Q B_1 c_0^2 w_1^2\right], \quad (2.22)$$

where Q is given by (2.15).
Since $h(z)$ is analytic and bounded in $\mathbb{U}$, hence by ([17], p. 172), we have

$$|c_0| \le 1 \quad \text{and} \quad |c_n| = 1 - |c_0|^2 \le 1 \text{ for } n > 0. \quad (2.23)$$

By using this fact and $|w_1| \le 1$, we get from (2.20), (2.21), (2.22) and (2.23) we obtain

$$|a_2| \le \frac{|\gamma| B_1}{(1+\alpha)(1+\lambda)}, \quad (2.24)$$

$$|a_3| \le \frac{|\gamma| B_1}{2(1+2\alpha)(1+2\lambda)}\left\{1 + \left|w_2 - \left(-\frac{(1+3\alpha)\gamma}{(1+\alpha)^2} B_1 c_0 - \frac{B_2}{B_1}\right) w_1^2\right|\right\}, \quad (2.25)$$

and

$$|a_3 - \mu a_2^2| \quad \le \tfrac{|\gamma| B_1}{2(1+2\alpha)(1+2\lambda)}\left[1 + \left|w_2 - \left(Q B_1 c_0 - \tfrac{B_2}{B_1}\right) w_1^2\right|\right]. \quad (2.26)$$

Case-I: If $c_0 = 0$, then (2.22) gives

$$|a_3 - \mu a_2^2| \le \frac{|\gamma| B_1}{2(1+2\alpha)(1+2\lambda)}. \quad (2.27)$$

Case-II: If $c_0 \ne 0$, then by applying the Lemma 1.1 to

$$\left|w_2 - \left(Q B_1 c_0 - \frac{B_2}{B_1}\right) w_1^2\right|, \quad (2.28)$$

we get from (2.26)

$$|a_3 - \mu a_2^2| \le \frac{|\gamma| B_1}{2(1+2\alpha)(1+2\lambda)}\left[1 + \max\left\{1, \left(|Q| B_1 + \frac{|B_2|}{B_1}\right)\right\}\right]. \quad (2.29)$$

The required result (2.14) follows from (2.27) and (2.29). In a similar manner we can prove the required assertion (2.13). The result is sharp for the function $f(z)$ given by

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)}+\alpha\left(1+\frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)}\right)-1\right]=\phi(z)-1,$$

or

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)}+\alpha\left(1+\frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)}\right)-1\right]=\phi(z^2)-1.$$

This completes the proof of Theorem 2.1. □

Putting $\gamma=1$, $\alpha=0$ and $\lambda=0$ in Theorem 2.1, we get the following sharp results for the class $\mathcal{S}_q^*(\phi)$.

Corollary 2.1. *Let $f\in\mathcal{A}$ of the form* (1.1) *belongs to the class $\mathcal{S}_q^*(\phi)$, then*

$$|a_2|\leq B_1$$

and for any complex number μ,

$$|a_3-\mu a_2^2|\leq\frac{B_1}{2}\left[1+\max\left\{1,|1-2\mu|B_1+\frac{|B_2|}{B_1}\right\}\right].$$

The result is sharp.

Putting $\gamma=1$, $\alpha=0$ and $\lambda=1$ in Theorem 2.1, we get the following sharp results for the class $\mathcal{C}_q(\phi)$.

Corollary 2.2. *Let $f\in\mathcal{A}$ of the form* (1.1) *belongs to the class $\mathcal{C}_q(\phi)$, then*

$$|a_2|\leq\frac{B_1}{2}$$

and for any complex number μ,

$$|a_3-\mu a_2^2|\leq\frac{B_1}{6}\left[1+\max\left\{1,\left(\left|1-\frac{3\mu}{2}\right|B_1+\frac{|B_2|}{B_1}\right)\right\}\right].$$

The result is sharp.

Remark 2.1. The Corollary 2.1 and Corollary 2.2 are due to the results obtained by Mohd and Darus [14].

The next theorem gives the result based on majorization.

Theorem 2.2. *Let $0\neq\gamma\in\mathbb{C}$, $\alpha\geq 0$ and $0\leq\lambda\leq 1$. If $f\in\mathcal{A}$ of the form* (1.1) *satisfies*

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)}+\alpha\left(1+\frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)}\right)-1\right]\ll(\phi(z)-1),\ z\in\mathbb{U},\tag{2.30}$$

then

$$|a_2| \leq \frac{|\gamma|B_1}{(1+\alpha)(1+\lambda)}, \tag{2.31}$$

$$|a_3| \leq \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}\left[1+\frac{(1+3\alpha)|\gamma|}{(1+\alpha)^2}B_1+\frac{|B_2|}{B_1}\right] \tag{2.32}$$

and for any complex number μ*,*

$$|a_3-\mu a_2^2| \leq \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}\left[1+|Q|B_1+\frac{|B_2|}{B_1}\right], \tag{2.33}$$

where Q *is given by* (2.15)*. The result is sharp.*

Proof. Let us assume that (2.30) holds. Then from the definition of majorization, there exists an analytic function $h(z)$ such that

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)}+\alpha\left(1+\frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)}\right)-1\right]=h(z)(\phi(z)-1). \tag{2.34}$$

Following similar steps as in the Theorem 2.1, and by setting $w(z)=z$, that is, for $w_1=1$, $w_n=0$, $n\geq 2$, we obtain

$$a_2=\frac{\gamma B_1 c_0}{(1+\alpha)(1+\lambda)},$$

which gives on use of the fact $c_n \leq 1$, for $n>0$,

$$|a_2| \leq \frac{|\gamma|B_1}{(1+\alpha)(1+\lambda)},$$

$$a_3=\tfrac{\gamma}{2(1+2\alpha)(1+2\lambda)}\left[B_1c_1+c_0\left(B_2+\tfrac{(1+3\alpha)\gamma}{(1+\alpha)^2}B_1^2c_0\right)\right]. \tag{2.35}$$

Thus for any complex number μ, we have

$$a_3-\mu a_2^2=\frac{\gamma B_1}{2(1+2\alpha)(1+2\lambda)}\left[c_1+c_0\left(\frac{B_2}{B_1}\right)-QB_1c_0^2\right]. \tag{2.36}$$

Following similar steps in Theorem2.1 we get the following from (2.36):
for $c_0=0$,

$$|a_3-\mu a_2^2| \leq \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}, \tag{2.37}$$

and for $c_0 \neq 0$

$$|a_3-\mu a_2^2| \leq \frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}\left[1+\frac{|B_2|}{B_1}+|Q|B_1\right]. \tag{2.38}$$

Thus, the assertion (2.33) of Theorem 2.2 follows from (2.37) and (2.38). Following the above steps we can prove the assertion (2.32) of Theorem 2.2. The result is sharp for the function

$$\frac{1}{\gamma}\left[(1-\alpha)\frac{z\mathcal{F}_\lambda'(z)}{\mathcal{F}_\lambda(z)}+\alpha\left(1+\frac{z\mathcal{F}_\lambda''(z)}{\mathcal{F}_\lambda'(z)}\right)-1\right]=\phi(z)-1, \quad z\in\mathbb{U},$$

which completes the proof of the Theorem 2.2. □

For $h(z)=1$, that is, for $c_0=1$ and $c_n=0, n\geq 1$, we have the following theorem:

Theorem 2.3. *Let* $0\neq\gamma\in\mathbb{C}$, $\alpha\geq 0$ *and* $0\leq\lambda\leq 1$. *If* $f\in\mathcal{A}$ *of the form* (1.1) *belongs* $\mathcal{M}^\alpha(\gamma,\lambda,\phi)$, *then*

$$|a_2|\leq\frac{|\gamma|B_1}{(1+\alpha)(1+\lambda)}, \tag{2.39}$$

$$|a_3|\leq\frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}\left[\max\left\{1,\left(\frac{(1+3\alpha)|\gamma|}{(1+\alpha)^2}B_1+\frac{|B_2|}{B_1}\right)\right\}\right] \tag{2.40}$$

and for any complex number μ,

$$|a_3-\mu a_2^2|\leq\frac{|\gamma|B_1}{2(1+2\alpha)(1+2\lambda)}\left[\max\left\{1,\left(|Q|B_1+\frac{|B_2|}{B_1}\right)\right\}\right], \tag{2.41}$$

where Q *is given by* (2.15). *The result is sharp.*

Proof. Proof is similar to Theorem 2.1. □

Remark 2.2. For $\gamma=1$ and $\lambda=0$, the Theorem 2.3 due to the result in [14] and [2] for $k=1$.

Conclusion: In this paper we have introduced a new subclass of univalent functions and obtained sharp coefficient estimates.

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Some Strongly Almost Summable Sequence Spaces

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Abstract: In the present paper we introduce some strongly almost summable sequence spaces using ideal convergence and Musielak-Orlicz function $\mathcal{M} = (M_k)$ in n-normed spaces. We examine some topological properties of the resulting sequence spaces.

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1. Introduction and preliminaries

Mursaleen and Noman [18] introduced the notion of λ-convergent and λ-bounded sequences as follows:
Let $\lambda = (\lambda_k)_{k=1}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity i.e.

$$0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty$$

and said that a sequence $x = (x_k) \in w$ is λ-convergent to the number L, called the λ-limit of x if $\Lambda_m(x) \longrightarrow L$ as $m \to \infty$, where

$$\lambda_m(x) = \frac{1}{\lambda_m}\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1})x_k.$$

The sequence $x = (x_k) \in w$ is λ-bounded if $\sup_m |\Lambda_m(x)| < \infty$. It is well known [18] that if $\lim_m x_m = a$ in the ordinary sense of convergence, then

$$\lim_m \left(\frac{1}{\lambda_m}\left(\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1})|x_k - a|\right)\right) = 0.$$

This implies that

$$\lim_m |\Lambda_m(x) - a| = \lim_m |\frac{1}{\lambda_m}\sum_{k=1}^{m}(\lambda_k - \lambda_{k-1})(x_k - a)| = 0,$$

which yields that $\lim_m \Lambda_m(x) = a$ and hence $x = (x_k) \in w$ is λ-convergent to a.
The concept of 2-normed spaces was initially developed by Gähler [6] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [15]. Since then, many others have studied this concept and obtained various results, see Gunawan ([8, 9]) and Gunawan and Mashadi [10] and references therein. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;
2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;
3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \; ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{K}$, and
4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called a n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called a n-normed space over the field $\mathbb{K}$.
For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $||x_1, x_2, \cdots, x_n||_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{\substack{k \to \infty \\ p \to \infty}} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to define the following sequence space,

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\},$$

which is called as an Orlicz sequence space. Also ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

Also, it was shown in [13] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p(p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see ([14, 23]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \ldots$$

is called the complementary function of the Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$ for all $x \in X$,
2. $p(-x) = p(x)$ for all $x \in X$,
3. $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [30], Theorem 10.4.2, p. 183). For more details about sequence spaces (see [16, 17, 19, 20, 21, 22, 24, 25, 26, 27, 29]) and reference therein.
A sequence space E is said to be solid (or normal) if $(x_k) \in E$ implies $(\alpha_k x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ and for all $k \in \mathbb{N}$.
The notion of ideal convergence was introduced first by P. Kostyrko [11] as a generalization of statistical convergence which was further studied in topological spaces (see [2]). More applications of ideals can be seen in [2, 3].
A linear functional $\mathcal{L}$ on ℓ_∞ is said to be a Banach limit see [1] if it has the properties:

1. $\mathcal{L}(x) \geq 0$ if $x \geq 0$ (i.e. $x_n \geq 0$ for all n),
2. $\mathcal{L}(e) = 1$, where $e = (1, 1, \ldots)$,
3. $\mathcal{L}(Dx) = \mathcal{L}(x)$,

where the shift operator D is defined by $(Dx_n) = (x_{n+1})$.
Let $\mathfrak{B}$ be the set of all Banach limits on ℓ_∞. A sequence x is said to be almost convergent to a number L if $\mathcal{L}(x) = L$ for all $\mathcal{L} \in \mathfrak{B}$. Lorentz [12] has shown that x is almost convergent to L if and only if

$$t_{km} = t_{km}(x) = \frac{x_m + x_{m+1} + \cdots + x_{m+k}}{k+1} \to L \text{ as } k \to \infty, \text{ uniformly in } m.$$

Recently a lot of activities have started to study sumability, sequence spaces and related topics in these non linear spaces see [4, 28]. In particular Sahiner [28] combined these two concepts and investigated ideal sumability in these spaces and introduced certain sequence spaces using 2-norm.
We continue in this direction and by using Musielak-Orlicz function, generalized sequences and also ideals we introduce I-convergence of generalized sequences with respect to Musielak-Orlicz function in n-normed spaces.

Let $(X, ||.||)$ be a normed space. Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X is called statistically convergent to $x \in X$ if the set $A(\epsilon) = \left\{ n \in \mathbb{N} : ||x_n - x|| \geq \epsilon \right\}$ has natural density zero for each $\epsilon > 0$.
A family $\mathcal{I} \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Yif

1. $\phi \in \mathcal{I}$;
2. $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$;
3. $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ (see [6]).

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \epsilon\Big\}$ belongs to $\mathcal{I}$ (see [11]).
Let I be an admissible ideal of $\mathbb{N}$, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space. Let $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers. By $S(n-X)$ we denote the space of all sequences defined over $(X, ||\cdot, \cdots, \cdot||)$. We define the following sequence spaces in this paper:
$\hat{w}^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in S(n-X) : \forall\ \epsilon > 0,\ \Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x) - L)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I$$
$$\text{for some}\ \ \rho > 0,\ \ L \in X\ \ \text{and}\ \ \ z_1, \cdots, z_{n-1} \in X\Big\},$$

$\hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in S(n-X) : \forall\ \epsilon > 0,\ \Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I$$
$$\text{for some}\ \ \rho > 0,\ \ \text{and}\ \ \ z_1, \cdots, z_{n-1} \in X\Big\},$$

$\hat{w}_\infty(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in S(n-X) : \exists\ K > 0\ \ \text{such that}\ \ \sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq K$$
$$\text{for some}\ \ \rho > 0,\ \ \text{and}\ \ z_1, \cdots, z_{n-1} \in X\Big\},$$

$\hat{w}_\infty^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||) =$

$$\Big\{x = (x_k) \in S(n-X) : \exists\ K > 0\ \ \text{such that}$$

$$\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq K\Big\} \in I$$

$$\text{for some } \rho > 0, \text{ and } z_1, \cdots, z_{n-1} \in X\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.
The main aim of this paper is to study some topological properties and inclusion relations between the above defined sequence spaces.

2. Main results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and I be an admissible ideal of $\mathbb{N}$. Then $\hat{w}^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$, $\hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$, $\hat{w}_\infty(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$ and $\hat{w}_\infty^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$ are linear spaces.*

Proof. Let $x, y \in \hat{w}^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$ and $\alpha, \beta \in \mathbb{C}$. So

$$\Big\{\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x) - L)}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I \quad \text{for some } \rho_1 > 0,$$

$$L \in X \text{ and } \quad z_1, \cdots, z_{n-1} \in X\Big\}$$

and

$$\Big\{\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(y) - L)}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I \text{ for some } \rho_2 > 0,$$

$$L \in X \text{ and } \quad z_1, \cdots, z_{n-1} \in X\Big\}.$$

Since $||\cdot, \cdots, \cdot||$ is a n-norm, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and so by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(\alpha x+\beta y)-L)}{|\alpha|\rho_1+|\beta|\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\leq\ D\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}M_k\Big(||\frac{t_{km}(\Lambda_k(x)-L)}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&+\ D\frac{1}{n}\sum_{k=1}^{n}\Big[\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}M_k\Big(||\frac{t_{km}(\Lambda_k(y)-L)}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&\leq\ DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x)-L)}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\\
&+\ DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(y)-L)}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k},
\end{aligned}$$

where $F=\max\Big[1,\Big(\frac{|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H,\Big(\frac{|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H\Big]$. From the above inequality, we get

$$\Big\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(\alpha x+\beta y)-L)}{|\alpha|\rho_1+|\beta|\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\geq\epsilon\Big\}$$

$$\begin{aligned}
&\subseteq\ \Big\{n\in\mathbb{N}:DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x)-L)}{\rho_1},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\geq\frac{\epsilon}{2}\Big\}\\
&\cup\ \Big\{n\in\mathbb{N}:DF\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(y)-L)}{\rho_2},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\geq\frac{\epsilon}{2}\Big\}.
\end{aligned}$$

Two sets on the right hand side belong to I and this completes the proof. Similarly, we can prove that $\hat{w}_0^I(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$, $\hat{w}_\infty(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$ and $\hat{w}_\infty^I(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$ are linear spaces.

Theorem 2.2. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function, $p=(p_k)$ be a bounded sequence of positive real numbers. For any fixed $n\in\mathbb{N}$, $\hat{w}_\infty(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$ is a paranormed space with the paranorm defined by*

$$g(x)=\inf\Big\{\rho^{\frac{p_n}{H}}:\rho>0\quad\textit{is such that}\ \sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\leq 1,\quad\forall z_1,\cdots,z_{n-1}\in X\Big\}.$$

Proof. It is clear that $g(x)=g(-x)$. Since $M_k(0)=0$, we get $\inf\{\rho^{\frac{p_n}{H}}\}=0$ for $x=0$ therefore, $g(0)=0$. Let us take $x,y\in\hat{w}_\infty(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$. Let

$$B(x)=\Big\{\rho^{\frac{p_n}{H}}:\rho>0,\sup_k\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\leq 1,$$

$$\forall z_1, \cdots, z_{n-1} \in X\Big\},$$

$$B(y) = \Big\{\rho^{\frac{p_n}{H}} : \rho > 0, : \sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(y))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq 1,$$

$$\forall z_1, \cdots, z_{n-1} \in X\Big\}.$$

Let $\rho_1 \in B(x)$ and $\rho_2 \in B(y)$. If $\rho = \rho_1 + \rho_2$, then we have
$\sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x+y))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]$

$$\begin{aligned} &\leq \frac{\rho_1}{\rho_1+\rho_2}\sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big] \\ &+ \frac{\rho_2}{\rho_1+\rho_2}\sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(y))}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]. \end{aligned}$$

Thus $\sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x+y))}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \leq 1$ and

$$\begin{aligned} g(x+y) &\leq \inf\Big\{(\rho_1+\rho_2)^{\frac{p_n}{H}} : \rho_1 \in B(x),\ \rho_2 \in B(y)\Big\} \\ &\leq \inf\Big\{\rho_1^{\frac{p_n}{H}} : \rho_1 \in B(x)\Big\} + \inf\Big\{\rho_2^{\frac{p_n}{H}} : \rho_2 \in B(y)\Big\} \\ &= g(x) + g(y). \end{aligned}$$

Let $\sigma^m \to \sigma$ where $\sigma, \sigma^m \in \mathbb{C}$ and let $g(x^m - x) \to 0$ as $m \to \infty$. We have to show that $g(\sigma^m x^m - \sigma_x) \to 0$ as $m \to \infty$. Let

$$B(x^m) = \Big\{\rho_m^{\frac{p_n}{H}} : \rho_m > 0, \sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x^m))}{\rho_m}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq 1,$$

$$\forall z_1, \cdots, z_{n-1} \in X\Big\},$$

$$B(x^m - x) = \Big\{\rho_m'^{\frac{p_n}{H}} : \rho_m' > 0, \sup_k \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x^m - x))}{\rho_m'}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq 1,$$

$$\forall z_1, \cdots, z_{n-1} \in X\Big\}.$$

If $\rho_m \in B(x^m)$ and $\rho_m' \in B(x^m - x)$ then we observe that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[M_k||\frac{t_{km}(\sigma^m\Lambda_k(x^m)-\sigma\Lambda_k(x))}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|},z_1,\cdots,z_{n-1}||$$

$$\begin{aligned}
&\le \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\sigma^m\Lambda_k(x^m)-\sigma\Lambda_k(x^m))}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|},z_1,\cdots,z_{n-1}||\\
&+ ||\frac{t_{km}(\sigma\Lambda_k(x^m)-\sigma\Lambda_k(x))}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|},z_1,\cdots,z_{n-1}||\Big)\Big]\\
&\le \frac{|\sigma^m-\sigma|\rho_m}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|}\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x^m))}{\rho_m},z_1,\cdots,z_{n-1}||\Big)\Big]\\
&+ \frac{|\sigma|\rho_m^{'}}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|}\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x^m)-\Lambda_k(x))}{\rho_m^{'}},z_1,\cdots,z_{n-1}||\Big)\Big].
\end{aligned}$$

From the above inequality, it follows that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\sigma^m\Lambda_k(x^m)-\sigma\Lambda_k(x))}{\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\le 1$$

and consequently,

$$\begin{aligned}
g(\sigma^m x^m-\sigma x) &\le \inf\Big\{\Big(\rho_m|\sigma^m-\sigma|+\rho_m^{'}|\sigma|\Big)^{\frac{p_n}{H}}:\rho_m\in B(x^m),\rho_m^{'}\in B(x^m-x)\Big\}\\
&\le (|\sigma^m-\sigma|)^{\frac{p_n}{H}}\inf\Big\{\rho^{\frac{p_n}{H}}:\rho_m\in B(x^m)\Big\}\\
&+ (|\sigma|)^{\frac{p_n}{H}}\inf\Big\{(\rho_m^{'})^{\frac{p_n}{H}}:\rho_m^{'}\in B(x^m-x)\Big\}\longrightarrow 0 \text{ as } m\longrightarrow\infty.
\end{aligned}$$

This completes the proof.

Theorem 2.3. *Let $\mathcal{M},\mathcal{M}',\mathcal{M}''$ are Musielak-Orlicz functions. Then we have*
(i) $\hat{w}_0^I(\mathcal{M}',\Lambda,p,||\cdot,\cdots,\cdot||)\subseteq\hat{w}_0^I(\mathcal{M}\circ\mathcal{M}',\Lambda,p,||\cdot,\cdots,\cdot||)$ *provided (p_k) is such that*

$$H_0=\inf p_k>0.$$

(ii) $\hat{w}_0^I(\mathcal{M}',\Lambda,p,||\cdot,\cdots,\cdot||)\cap\hat{w}_0^I(\mathcal{M}'',\Lambda,p,||\cdot,\cdots,\cdot||)\subseteq\hat{w}_0^I(\mathcal{M}'+\mathcal{M}'',\Lambda,p,||\cdot,\cdots,\cdot||)$.

Proof. (i) For given $\epsilon>0$, first choose $\epsilon_0>0$ such that $\max\{\epsilon_0^H,\epsilon_0^{H_0}\}<\epsilon$. Since (M_k) is continuous, choose $0<\delta<1$ such that $0<t<\delta$, this implies that $M_k(t)<\epsilon_0$. Let $x\in\hat{w}_0^I(\mathcal{M},\Lambda,p,||\cdot,\cdots,\cdot||)$. Now from the definition

$$B(\delta)=\Big\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\Big[M_k'\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}\ge\delta^H\Big\}\in I.$$

Thus if $n\notin B(\delta)$ then

$$\frac{1}{n}\sum_{k=1}^{n}\Big[M_k'\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho},z_1,\cdots,z_{n-1}||\Big)\Big]^{p_k}<\delta^H$$

$$\Longrightarrow \sum_{k=1}^{n}\Big[M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < n\delta^H$$

$$\Longrightarrow \Big[M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \delta^H \text{ for all } k, m = 1, 2, 3, \ldots$$

$$\Longrightarrow \Big[M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] < \delta \text{ for all } k, m = 1, 2, 3, \ldots.$$

Hence from above and using the continuity of $\mathcal{M} = (M_k)$, we have

$$\Big[M_k\Big(M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big] < \epsilon_0 \ \forall \ k, m = 1, 2, 3, \ldots ,$$

which consequently implies that

$$\sum_{k=1}^{n}\Big[M_k\Big(M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} < \max\{\epsilon_0^H, \epsilon_0^{H_0}\} < \epsilon.$$

$$\text{Thus } \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} < \epsilon.$$

This shows that

$$\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \geq \epsilon\Big\} \subset B(\delta)$$

and so belongs to I. This proves the result.

(ii) Let $(x_k) \in \hat{w}_0^I(\mathcal{M}', \Lambda, p, ||\cdot, \cdots, \cdot||) \cap \hat{w}_0^I(\mathcal{M}'', \Lambda, p, ||\cdot, \cdots, \cdot||)$. Then the fact

$$\frac{1}{n}\Big[(M'_k + M''_k)\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq$$

$$D\frac{1}{n}\Big[M'_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} + D\frac{1}{n}\Big[M''_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

gives the result.

Theorem 2.4. *The sequence spaces* $\hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$ *and* $\hat{w}_\infty^I(\mathcal{M}', \Lambda, p, ||\cdot, \cdots, \cdot||)$ *are solid.*

Proof. Let $x \in \hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$, let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then we have

$$\Big\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(\alpha_k x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big\} \subset$$

$$\Big\{n \in \mathbb{N} : \frac{C}{n}\sum_{k=1}^{n}\Big[M_k\Big(||\frac{t_{km}(\Lambda_k(x))}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I,$$

where $C = \max\{1, |\alpha_k|^H\}$. Hence $(\alpha_k x) \in \hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$ for all sequences of scalars α_k with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$ whenever $(x_k) \in \hat{w}_0^I(\mathcal{M}, \Lambda, p, ||\cdot, \cdots, \cdot||)$. Similarly, we can prove that $\hat{w}_\infty^I(\mathcal{M}', \Lambda, p, ||\cdot, \cdots, \cdot||)$ is also solid.

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