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Solvability of a Quadratic Integral Equation of Fredholm Type Via a Modified Argument

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Abstract: This article concerns with the existence of solutions of the a quadratic integral equation of Fredholm type with a modified argument,

$$x(t) = p(t) + (Fx)(t) \int_0^1 k(t,\tau)x(q(\tau))d\tau,$$

where p, k are functions and F is an operator satisfying the given conditions. Using the properties of the Hölder spaces and the classical Schauder fixed point theorem, we obtain the existence of solutions of the equation under certain assumptions. Also, we present two concrete examples in which our result can be applied.

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Keywords and Phrases: Fredholm equation; Hölder condition; Schauder fixed point theorem.

1. Introduction

Integral equations arise from naturally in many applications in describing numerous real world problems (see, for instance, the books [2, 3] and references therein). Quadratic integral equations arise naturally in applications of real world problems. For example, problems in the theory of radiative transfer in the theory of neutron transport and in the kinetic theory of gases lead to the quadratic equation [12, 20]. There are many interesting existence results for all kinds of quadratic integral equations, one can refer to [6, 1].

The study of differential equations with a modified arguments arise in a wide variety of scientific and technical application, including the modelling of problems from the natural and social sciences such as physics, biological and economics sciences. A special class of these differential equations have linear modifications of their arguments, and have been studied by several authors, [7] - [23].

Recently, Banaś and Nalepa [7] have studied the space of real functions defined on a given bounded metric space and having the growths tempered by a given modulus of continuity, and derive the existence theorem in the space of functions satisfying the Hölder condition for some quadratic integral equations of Fredholm type

$$x(t) = p(t) + x(t) \int_a^b k(t, \tau)\ x(\tau) d\tau. \tag{1.1}$$

Further, Caballero et al. [9] have studied the solvability of the following quadratic integral equation of Fredholm type

$$x(t) = p(t) + x(t) \int_0^1 k(t, \tau)\ x(q(\tau)) d\tau \tag{1.2}$$

in Hölder spaces. The purpose of this paper is to investigate the existence of solutions of the following integral equation of Fredholm type with a modified argument in Hölder spaces

$$x(t) = p(t) + (Fx)(t) \int_0^1 k(t, \tau)\ x(q(\tau)) d\tau, \quad t \in I = [0, 1] \tag{1.3}$$

where p, k, q and F are functions satisfying the given conditions. To do this, we will use a recent result about the relative compactness in Hölder spaces and the classical Schauder fixed point theorem.

Notice that equation (1.1) in [9] is a particular case of (1.3), for $(Fx)(\tau) = x(\tau)$. The obtained result in this paper is more general than the result in [9].

2. Preliminaries

Let we introduce notations, definitions and theorems which are used throughout this paper.

By $C[a, b]$, we denote the space of continuous functions on $[a, b]$ equipped with usually the supremum norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in [a, b]\}$$

for $x \in C[a, b]$. For a fixed α with $0 < \alpha \leqslant 1$, we write $H_\alpha[a, b]$ to denote the set of all the real valued functions x defined on $[a, b]$ and satisfying the Hölder condition with α, that is, there exists a constant H such that the inequality

$$|x(t) - x(s)| \leqslant H|t - s|^\alpha \tag{2.1}$$

holds for all $t, s \in [a, b]$. One can easily seen that $H_\alpha[a, b]$ is a linear subspaces of $C[a, b]$. In the sequel, for $x \in H_\alpha[a, b]$, by H_x^α we will denote the least possible constant for which inequality (2.1) is satisfied. Rather, we put

$$H_x^\alpha = \sup\left\{\frac{|x(t) - x(s)|}{|t - s|^\alpha} : t, s \in [a, b], t \neq s\right\}. \tag{2.2}$$

The space $H_\alpha[a, b]$ with $0 < \alpha \leqslant 1$ can be equipped with the norm:

$$\|x\|_\alpha = |x(a)| + \sup\left\{\frac{|x(t) - x(s)|}{|t - s|^\alpha} : t, s \in [a, b], t \neq s\right\} \tag{2.3}$$

for $x \in H_\alpha[a, b]$. In [7], the authors proved that $(H_\alpha[a, b], \|\cdot\|_\alpha)$ with $0 < \alpha \leqslant 1$ is a Banach space. The following lemmas in [7] present some results related to the Hölder spaces and norm.

Lemma 2.1. *For $0 < \alpha \leqslant 1$ and $x \in H_\alpha[a, b]$, the following inequality is satisfied*

$$\|x\|_\infty \leqslant \max\{1, (b - a)^\alpha\} \|x\|_\alpha.$$

In particular, the inequality $\|x\|_\infty \leqslant \|x\|_\alpha$ holds, for $a = 0$ and $b = 1$.

Lemma 2.2. *For $0 < \alpha < \gamma \leqslant 1$, we have*

$$H_\gamma[a, b] \subset H_\alpha[a, b] \subset C[a, b].$$

Moreover, for $x \in H_\gamma[a, b]$ the following inequality holds

$$\|x\|_\alpha \leqslant \max\left\{1, (b - a)^{\gamma-\alpha}\right\} \|x\|_\gamma.$$

In particular, the inequality $\|x\|_\infty \leqslant \|x\|_\alpha \leqslant \|x\|_\gamma$ is satisfied for $a = 0$ and $b = 1$.

Now we present the important theorem which is the sufficient condition for relative compactness in the spaces $H_\alpha[a, b]$ with $0 < \alpha \leqslant 1$.

Theorem 2.3. *[9] Suppose that $0 < \alpha < \beta \leqslant 1$ and that A is a bounded subset of $H_\beta[a, b]$ (this means that $\|x\|_\beta \leqslant M$ for certain constant $M > 0$, for all $x \in A$) then A is a relatively compact subset of $H_\alpha[a, b]$.*

Lemma 2.4. *[9] Suppose that $0 < \alpha < \beta \leqslant 1$ and by B_r^β we denote the closed ball centered at θ with radius r in the space $H_\beta[a, b]$, i.e., $B_r^\beta = \{x \in H_\beta[a, b] : \|x\|_\beta \leqslant r\}$. Then B_r^β is a closed subset of $H_\alpha[a, b]$.*

Corollary 2.5. *Suppose that $0 < \alpha < \beta \leqslant 1$ then B_r^β is a compact subset of the space $H_\alpha[a, b]$, [9].*

Theorem 2.6 (Schauder's fixed point theorem)**.** *Let L be a nonempty, convex, and compact subset of a Banach space $(X, \|\cdot\|)$ and let $T : L \to L$ be a continuity mapping. Then T has at least one fixed point in L, [24].*

3. Main Result

In this section, we will study the solvability of the equation (1.3) in the space $H_\alpha[0,1]$ $(0 < \alpha \leqslant 1)$. We will use the following assumptions:

(i) $p \in H_\beta[0,1]$, $0 < \beta \leqslant 1$.

(ii) $k : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function such that it satisfies the Hölder condition with exponent β with respect to the first variable, that is, there exists a constant $k_\beta > 0$ such that:

$$|k(t,\tau) - k(s,\tau)| \leqslant k_\beta |t-s|^\beta,$$

for any $t, s, \tau \in [0,1]$.

(iii) $q : [0,1] \to [0,1]$ is a measurable function.

(iv) The operator $F : H_\beta[0,1] \to H_\beta[0,1]$ is continuous with respect to the norm $\|\cdot\|_\alpha$ for $0 < \alpha < \beta \leqslant 1$ and there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+ = [0,\infty)$ which is non-decreasing such that it holds the inequality

$$\|Fx\|_\beta \leqslant f(\|x\|_\beta),$$

for any $x \in H_\beta[0,1]$.

(v) There exists a positive solution r_0 of the inequality

$$\|p\|_\beta + (2K + k_\beta) r f(r) \leqslant r,$$

where K is a constant is satisfying the following inequality,

$$K = \sup\left\{ \int_0^1 |k(t,\tau)| d\tau : t \in [0,1] \right\}.$$

Theorem 3.1. *Under the assumptions (i)-(v), Equation (1.3) has at least one solution belonging to the space $H_\alpha[0,1]$.*

Proof. Consider the operator T below that defined on the space $H_\beta[0,1]$ by

$$(Tx)(t) = p(t) + (Fx)(t) \int_0^1 k(t,\tau) x(q(\tau)) d\tau, \quad t \in [0,1].$$

We will firstly prove that T transforms the space $H_\beta[0,1]$ into itself. For arbitrarily fixed $x \in H_\beta[0,1]$ and $t, s \in [0,1]$ with $(t \neq s)$, taking into account assumptions

(i), (ii) and (iii), we obtain

$$\frac{|(Tx)(t)-(Tx)(s)|}{|t-s|^\beta}$$

$$= \frac{\left|p(t)+(Fx)(t)\int_0^1 k(t,\tau)x(q(\tau))\,d\tau - p(s) - (Fx)(s)\int_0^1 k(s,\tau)x(q(\tau))\,d\tau\right|}{|t-s|^\beta}$$

$$\leqslant \frac{1}{|t-s|^\beta}\Bigg[|p(t)-p(s)| + \Bigg|(Fx)(t)\int_0^1 k(t,\tau)x(q(\tau))\,d\tau$$

$$-(Fx)(s)\int_0^1 k(s,\tau)x(q(\tau))\,d\tau\Bigg|\Bigg]$$

$$\leqslant \frac{|p(t)-p(s)|}{|t-s|^\beta}$$

$$+\frac{1}{|t-s|^\beta}\left|(Fx)(t)\int_0^1 k(t,\tau)x(q(\tau))\,d\tau - (Fx)(s)\int_0^1 k(t,\tau)x(q(\tau))\,d\tau\right|$$

$$+\frac{1}{|t-s|^\beta}\left|(Fx)(s)\int_0^1 k(t,\tau)x(q(\tau))\,d\tau - (Fx)(s)\int_0^1 k(s,\tau)x(q(\tau))\,d\tau\right|$$

$$\leqslant \frac{|p(t)-p(s)|}{|t-s|^\beta} + \frac{|(Fx)(t)-(Fx)(s)|}{|t-s|^\beta}\int_0^1 |k(t,\tau)|\,|x(q(\tau))|\,d\tau$$

$$+\frac{|(Fx)(s)|\int_0^1 |k(t,\tau)-k(s,\tau)|\,|x(q(\tau))|\,d\tau}{|t-s|^\beta}$$

$$\leqslant \frac{|p(t)-p(s)|}{|t-s|^\beta} + \frac{|(Fx)(t)-(Fx)(s)|}{|t-s|^\beta}\|x\|_\infty\int_0^1 |k(t,\tau)|\,d\tau$$

$$+\frac{\|Fx\|_\infty\|x\|_\infty\int_0^1 |k(t,\tau)-k(s,\tau)|\,d\tau}{|t-s|^\beta}$$

$$\leqslant \frac{|p(t)-p(s)|}{|t-s|^\beta} + \frac{|(Fx)(t)-(Fx)(s)|}{|t-s|^\beta}\|x\|_\infty K + \frac{\|Fx\|_\infty\|x\|_\infty\int_0^1 k_\beta|t-s|^\beta\,d\tau}{|t-s|^\beta}$$

$$\leqslant H_p^\beta + H_{Fx}^\beta\|x\|_\infty K + \|Fx\|_\infty\|x\|_\infty k_\beta.$$

By using the facts that $\|x\|_\infty \leqslant \|x\|_\beta$ and $H_x^\beta \leqslant \|x\|_\beta$ concluded Lemma 2.1 and the definition $\|x\|_\beta$, respectively we infer that

$$\frac{|(Tx)(t)-(Tx)(s)|}{|t-s|^\beta} \leqslant H_p^\beta + (K+k_\beta)\|x\|_\beta\,\|Fx\|_\beta\,.$$

From this inequality, we have $Tx \in H_\beta[0,1]$. This proves that the operator T

maps the space $H_\beta[0,1]$ into itself. On the other hand we can write

$$\begin{aligned}
\|Tx\|_\beta &= |(Tx)(0)| + \sup\left\{\frac{|(Tx)(t)-(Tx)(s)|}{|t-s|^\beta} : t,s\in[0,1], t\neq s\right\} \\
&\leqslant |(Tx)(0)| + H_p^\beta + (K+k_\beta)\|x\|_\beta \|Fx\|_\beta \\
&\leqslant |p(0)| + |(Fx)(0)| \int_0^1 |k(0,\tau)|\, |x(q(\tau))|\, d\tau + H_p^\beta + (K+k_\beta)\|x\|_\beta \|Fx\|_\beta \\
&\leqslant \|p\|_\beta + \|Fx\|_\infty \|x\|_\infty \int_0^1 |k(0,\tau)|\, d\tau + (K+k_\beta)\|x\|_\beta \|Fx\|_\beta \\
&\leqslant \|p\|_\beta + K\|Fx\|_\beta \|x\|_\beta + (K+k_\beta)\|x\|_\beta \|Fx\|_\beta \\
&= \|p\|_\beta + (2K+k_\beta)\|x\|_\beta \|Fx\|_\beta \\
&\leqslant \|p\|_\beta + (2K+k_\beta)\|x\|_\beta f\left(\|x\|_\beta\right), \qquad (3.1)
\end{aligned}$$

for any $x \in H_\beta[0,1]$. So, if we take x in $B_{r_0}^\beta$ then by assumption (v) we get $Tx \in B_{r_0}^\beta$. As a result, it follows that T transforms the ball

$$B_{r_0}^\beta = \{x \in H_\beta[0,1] : \|x\|_\beta \leqslant r_0\}$$

into itself. That is,

$$T : B_{r_0}^\beta \to B_{r_0}^\beta.$$

Next, we will prove that the operator T is continuous on $B_{r_0}^\beta$, according to the induced norm by $\|\cdot\|_\alpha$, where $0<\alpha<\beta\leqslant 1$. To do this, let us take any fixed $y \in B_{r_0}^\beta$ and arbitrary $\varepsilon > 0$. Since the operator $F : H_\beta[0,1] \to H_\beta[0,1]$ is continuous on $H_\beta[0,1]$ with respect to the norm $\|\cdot\|_\alpha$, there exists $\delta > 0$ such that the inequality

$$\|Fx - Fy\|_\alpha < \frac{\varepsilon}{4(K+k_\beta)\, r_0}$$

is satisfied for all $x \in B_{r_0}^\beta$, such that $\|x-y\|_\alpha \leqslant \delta$ and

$$0 < \delta < \frac{\varepsilon}{2(2K+k_\beta)\, f(r_0)}.$$

Then, for any $x \in B_{r_0}^\beta$ and $t,s \in [0,1]$ with $t \neq s$ and $0 < \alpha \leqslant 1$ we have

$$\frac{|[(Tx)(t)-(Ty)(t)]-[(Tx)(s)-(Ty)(s)]|}{|t-s|^{\alpha}}$$

$$=\left|\frac{\left[(Fx)(t)\int_0^1 k(t,\tau)x(q(\tau))\,d\tau-(Fy)(t)\int_0^1 k(t,\tau)y(q(\tau))d\tau\right]}{|t-s|^{\alpha}}\right.$$

$$\left.-\ \frac{\left[(Fx)(s)\int_0^1 k(s,\tau)x(q(\tau))\,d\tau-(Fy)(s)\int_0^1 k(s,\tau)y(q(\tau))d\tau\right]}{|t-s|^{\alpha}}\right|$$

$$=\frac{1}{|t-s|^{\alpha}}\left|\left[(Fx)(t)\int_0^1 k(t,\tau)x(q(\tau))\,d\tau-(Fy)(t)\int_0^1 k(t,\tau)x(q(\tau))\,d\tau\right]\right.$$

$$+\left[(Fy)(t)\int_0^1 k(t,\tau)x(q(\tau))\,d\tau-(Fy)(t)\int_0^1 k(t,\tau)y(q(\tau))d\tau\right]$$

$$-\left[(Fx)(s)\int_0^1 k(s,\tau)x(q(\tau))\,d\tau-(Fy)(s)\int_0^1 k(s,\tau)x(q(\tau))\,d\tau\right]$$

$$\left.-\ \left[(Fy)(s)\int_0^1 k(s,\tau)x(q(\tau))\,d\tau-(Fy)(s)\int_0^1 k(s,\tau)y(q(\tau))d\tau\right]\right|$$

$$=\frac{1}{|t-s|^{\alpha}}\left|\left[[(Fx)(t)-(Fy)(t)]\int_0^1 k(t,\tau)x(q(\tau))\,d\tau\right]\right.$$

$$+\left[(Fy)(t)\int_0^1 k(t,\tau)\left[x(q(\tau))-y(q(\tau))\right]d\tau\right]$$

$$-\left[[(Fx)(s)-(Fy)(s)]\int_0^1 k(s,\tau)x(q(\tau))\,d\tau\right]$$

$$\left.-\ \left[(Fy)(s)\int_0^1 k(s,\tau)\left[x(q(\tau))-y(q(\tau))\right]d\tau\right]\right|$$

$$=\frac{1}{|t-s|^{\alpha}}\left|\{[(Fx)(t)-(Fy)(t)]-[(Fx)(s)-(Fy)(s)]\}\int_0^1 k(t,\tau)x(q(\tau))\,d\tau\right.$$

$$+\left[[(Fx)(s)-(Fy)(s)]\int_0^1 k(t,\tau)x(q(\tau))\,d\tau\right]$$

$$-\left[[(Fx)(s)-(Fy)(s)]\int_0^1 k(s,\tau)x(q(\tau))\,d\tau\right]$$

$$+\left[(Fy)(t)\int_0^1 k(t,\tau)\left[x(q(\tau))-y(q(\tau))\right]d\tau\right]$$

$$\left.-\ \left[(Fy)(s)\int_0^1 k(s,\tau)\left[x(q(\tau))-y(q(\tau))\right]d\tau\right]\right|$$

$$= \frac{1}{|t-s|^{\alpha}} \left| \{[(Fx)(t) - (Fy)(t)] - [(Fx)(s) - (Fy)(s)]\} \int_0^1 k(t,\tau) x(q(\tau))\, d\tau \right.$$

$$+ \left[[(Fx)(s) - (Fy)(s)] \int_0^1 (k(t,\tau) - k(s,\tau))\, x(q(\tau))\, d\tau \right]$$

$$+ \left[(Fy)(t) \int_0^1 k(t,\tau) [x(q(\tau)) - y(q(\tau))]\, d\tau \right]$$

$$\left. - \left[(Fy)(s) \int_0^1 k(s,\tau) [x(q(\tau)) - y(q(\tau))]\, d\tau \right] \right| .$$

From the last inequality it follows that

$$\frac{|[(Tx)(t) - (Ty)(t)] - [(Tx)(s) - (Ty)(s)]|}{|t-s|^{\alpha}}$$

$$\leqslant \frac{1}{|t-s|^{\alpha}} |[(Fx)(t) - (Fy)(t)] - [(Fx)(s) - (Fy)(s)]| \left| \int_0^1 k(t,\tau) x(q(\tau))\, d\tau \right|$$

$$+ \frac{1}{|t-s|^{\alpha}} |(Fx)(s) - (Fy)(s)| \left| \int_0^1 (k(t,\tau) - k(s,\tau))\, x(q(\tau))\, d\tau \right|$$

$$+ \frac{1}{|t-s|^{\alpha}} \left| (Fy)(t) \int_0^1 k(t,\tau) [x(q(\tau)) - y(q(\tau))]\, d\tau \right.$$

$$\left. - (Fy)(s) \int_0^1 k(s,\tau) [x(q(\tau)) - y(q(\tau))]\, d\tau \right|$$

$$\leqslant \frac{|[(Fx)(t) - (Fy)(t)] - [(Fx)(s) - (Fy)(s)]|}{|t-s|^{\alpha}} \|x\|_{\infty} \int_0^1 |k(t,\tau)|\, d\tau$$

$$+ |[(Fx)(s) - (Fy)(s)] - [(Fx)(0) - (Fy)(0)]|\, \|x\|_{\infty} \int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^{\alpha}} d\tau$$

$$+ |(Fx)(0) - (Fy)(0)|\, \|x\|_{\infty} \int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^{\alpha}} d\tau$$

$$+ \frac{1}{|t-s|^{\alpha}} \left| (Fy)(t) \int_0^1 k(t,\tau) [x(q(\tau)) - y(q(\tau))]\, d\tau \right.$$

$$\left. - (Fy)(s) \int_0^1 k(t,\tau) [x(q(\tau)) - y(q(\tau))]\, d\tau \right|$$

$$+ \frac{1}{|t-s|^{\alpha}} \left| (Fy)(s) \int_0^1 k(t,\tau) [x(q(\tau)) - y(q(\tau))]\, d\tau \right.$$

$$\left. - (Fy)(s) \int_0^1 k(s,\tau) [x(q(\tau)) - y(q(\tau))]\, d\tau \right|$$

$$\leqslant H^{\alpha}_{Fx-Fy} \|x\|_{\infty} K$$

$$+ \sup_{u,v\in[0,1]} |[(Fx)(u) - (Fy)(u)] - [(Fx)(v) - (Fy)(v)]| \|x\|_{\infty} \int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^{\alpha}} d\tau$$

$$+ |(Fx)(0) - (Fy)(0)| \|x\|_{\infty} \int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^{\alpha}} d\tau$$

$$+ \frac{|(Fy)(t) - (Fy)(s)|}{|t-s|^{\alpha}} \int_0^1 |k(t,\tau)| |x(q(\tau)) - y(q(\tau))| d\tau$$

$$+ |(Fy)(s)| \int_0^1 \frac{|k(t,\tau) - k(s,\tau)|}{|t-s|^{\alpha}} |x(q(\tau)) - y(q(\tau))| d\tau$$

$$\leqslant K \|x\|_{\infty} \|Fx - Fy\|_{\alpha}$$

$$+ \sup_{u,v\in[0,1]} |[(Fx)(u) - (Fy)(u)] - [(Fx)(v) - (Fy)(v)]| \|x\|_{\infty} \int_0^1 \frac{k_{\beta} |t-s|^{\beta}}{|t-s|^{\alpha}} d\tau$$

$$+ |(Fx)(0) - (Fy)(0)| \|x\|_{\infty} \int_0^1 \frac{k_{\beta} |t-s|^{\beta}}{|t-s|^{\alpha}} d\tau$$

$$+ \frac{|(Fy)(t) - (Fy)(s)|}{|t-s|^{\alpha}} \int_0^1 |k(t,\tau)| |x(q(\tau)) - y(q(\tau))| d\tau$$

$$+ |(Fy)(s)| \int_0^1 \frac{k_{\beta} |t-s|^{\beta}}{|t-s|^{\alpha}} |x(q(\tau)) - y(q(\tau))| d\tau.$$

In view of the inequalities $\|x\|_{\infty} \leqslant \|x\|_{\alpha}$, $H^{\beta}_{x} \leqslant \|x\|_{\alpha}$,we derive the following inequlities

$$\frac{|[(Tx)(t) - (Ty)(t)] - [(Tx)(s) - (Ty)(s)]|}{|t-s|^{\alpha}}$$

$$\leqslant K \|x\|_{\infty} \|Fx - Fy\|_{\alpha} + k_{\beta} \|x\|_{\infty} |t-s|^{\beta-\alpha} \cdot$$

$$\sup_{u,v\in[0,1],u\neq v} \left\{ \frac{|[(Fx)(u) - (Fy)(u)] - [(Fx)(v) - (Fy)(v)]|}{|u-v|^{\alpha}} |u-v|^{\alpha} \right\}$$

$$+ k_{\beta} \|x\|_{\infty} |t-s|^{\beta-\alpha} |(Fx)(0) - (Fy)(0)| + K H^{\alpha}_{Fy} \|x - y\|_{\infty}$$

$$+ k_{\beta} \|Fy\|_{\infty} \|x - y\|_{\infty} |t-s|^{\beta-\alpha}$$

$$\leqslant K \|x\|_{\beta} \|Fx - Fy\|_{\alpha} + 2k_{\beta} \|x\|_{\beta} \|Fx - Fy\|_{\alpha}$$

$$+ K \|Fy\|_{\alpha} \|x - y\|_{\alpha} + k_{\beta} \|Fy\|_{\alpha} \|x - y\|_{\alpha}$$

$$= (K + 2k_\beta) \|x\|_\beta \|Fx - Fy\|_\alpha$$
$$+ (K + k_\beta) \|Fy\|_\alpha \|x - y\|_\alpha . \qquad (3.2)$$

Since $\|y\|_\alpha \leqslant \|y\|_\beta \leqslant r_0$ (see Lemma 2.2) and from the assumption (iv) and (3.2) we deduce the following inequality

$$\frac{|[(Tx)(t) - (Ty)(t)] - [(Tx)(s) - (Ty)(s)]|}{|t - s|^\alpha}$$
$$\leqslant (K + 2k_\beta) \|x\|_\beta \|Fx - Fy\|_\alpha + (K + k_\beta) \|Fy\|_\beta \|x - y\|_\alpha$$
$$\leqslant (K + 2k_\beta) \|x\|_\beta \|Fx - Fy\|_\alpha + (K + k_\beta) f\left(\|y\|_\beta\right) \|x - y\|_\alpha$$
$$\leqslant (K + 2k_\beta) r_0 \|Fx - Fy\|_\alpha + (K + k_\beta) f(r_0) \delta. \qquad (3.3)$$

On the other hand,

$$|(Tx)(0) - (Ty)(0)| = \left|(Fx)(0) \int_0^1 k(0,\tau) x(q(\tau)) d\tau - (Fy)(0) \int_0^1 k(0,\tau) y(q(\tau)) d\tau\right|$$
$$\leqslant \left|(Fx)(0) \int_0^1 k(0,\tau) x(q(\tau)) d\tau - (Fx)(0) \int_0^1 k(0,\tau) y(q(\tau)) d\tau\right|$$
$$+ \left|(Fx)(0) \int_0^1 k(0,\tau) y(q(\tau)) d\tau - (Fy)(0) \int_0^1 k(0,\tau) y(q(\tau)) d\tau\right|$$
$$\leqslant |(Fx)(0)| \int_0^1 |k(0,\tau)| |x(q(\tau)) - y(q(\tau))| d\tau$$
$$+ |(Fx)(0) - (Fy)(0)| \int_0^1 |k(0,\tau)| |y(q(\tau))| d\tau$$

From the last inequality it follows that

$$|(Tx)(0) - (Ty)(0)| \leqslant K \|Fx\|_\infty \|x - y\|_\infty + K \|y\|_\infty \|Fx - Fy\|_\infty$$
$$\leqslant K \|Fx\|_\beta \|x - y\|_\alpha + K \|y\|_\beta \|Fx - Fy\|_\alpha$$
$$\leqslant K f\left(\|x\|_\beta\right) \|x - y\|_\alpha + K \|y\|_\beta \|Fx - Fy\|_\alpha$$
$$\leqslant K f(r_0) \delta + K r_0 \|Fx - Fy\|_\alpha . \qquad (3.4)$$

From (3.3) and (3.4), it follows that

$$\|Tx - Ty\|_\alpha$$
$$= |(Tx)(0) - (Ty)(0)| + H^\alpha_{Tx-Ty}$$
$$= |(Tx)(0) - (Ty)(0)| + \sup_{t,s\in[0,1], t\neq s} \left\{ \frac{|[(Tx)(t) - (Ty)(t)] - [(Tx)(s) - (Ty)(s)]|}{|t - s|^\alpha} \right\}$$

$$\leqslant Kf(r_0)\,\delta + Kr_0\,\|Fx - Fy\|_\alpha + (K + 2k_\beta)\,r_0\,\|Fx - Fy\|_\alpha + (K + k_\beta)\,f(r_0)\,\delta$$
$$= 2\,(K + k_\beta)\,r_0\,\|Fx - Fy\|_\alpha + (2K + k_\beta)\,f(r_0)\,\delta$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This show that the operator T is continuous at the point $y \in B^\beta_{r_0}$. We conclude that T is continuous on $B^\beta_{r_0}$ with respect to the norm $\|\cdot\|_\alpha$. In addition the set $B^\beta_{r_0}$ is compact subset of the space $H_\alpha[0,1]$ from [9] (see [9; the appendix at the p. 9]). Therefore, applying the classical Schauder fixed point theorem, we complete the proof. □

4. Examples

In this section, we provide an example illustrating the main results in the above.

Example 1. Let us consider the quadratic integral equation:

$$x(t) = \ln\left(\sqrt[4]{n\sin t + \hat{n}} + 1\right) + x^2(t)\int_0^1 \sqrt[3]{mt^3 + \tau}\,x\left(\frac{1}{\tau + 1}\right)d\tau \tag{4.1}$$

where $t \in [0,1]$ and $n, \hat{n}, m$ are the suitable non-negative constants.

Observe that (4.1) is a particular case of (1.3) if we put $p(t) = \ln\left(\sqrt[4]{n\sin t + \hat{n}} + 1\right)$, $k(t,\tau) = \sqrt[3]{mt^3 + \tau}$ and $q(\tau) = \frac{1}{\tau+1}$. The operator F defined by $(Fx)(t) = x^2(t)$ for all $t \in [0,1]$.

Since functions $s,\ h : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $s(t) = \ln(t+1)$, $h(t) = \sqrt[4]{t}$ are concav and $s(0) = 0$, $h(0) = 0$, then from Lemma 4.4 in [9] these functions are subadditive. If we consider the result of subadditivity and the inequalities $\ln x < x$ for $x > 0$ and $|\sin x - \sin y| \leqslant |x - y|$ for $x, y \in \mathbb{R}$, we can write

$$|p(t) - p(s)| = \left|\ln\left(\sqrt[4]{n\sin t + \hat{n}} + 1\right) - \ln\left(\sqrt[4]{n\sin s + \hat{n}} + 1\right)\right|$$
$$\leqslant \ln\left|\sqrt[4]{n\sin t + \hat{n}} - \sqrt[4]{n\sin s + \hat{n}}\right|$$
$$< \left|\sqrt[4]{n\sin t + \hat{n}} - \sqrt[4]{n\sin s + \hat{n}}\right|$$
$$\leqslant \left|\sqrt[4]{n\,|\sin t - \sin s|}\right|$$
$$\leqslant \sqrt[4]{n}\,|t - s|^{\frac{1}{4}}.$$

It means that $p \in H_{\frac{1}{4}}[0,1]$ and, moreover, $H_p^{\frac{1}{4}} = \sqrt[4]{n}$. We can take the constants α and β as $0 < \alpha < \frac{1}{4}$ and $\beta = \frac{1}{4}$. Therefore, assumption (i) of Theorem (3.1) is satisfied. Note that

$$\begin{aligned}\|p\|_{\frac{1}{4}} &= |p(0)| + \sup\left\{\frac{|p(t)-p(s)|}{|t-s|^{\frac{1}{4}}} : t,s\in[0,1], t\neq s\right\}\\ &= |p(0)| + H_p^{\frac{1}{4}} = \ln\left(\sqrt[4]{\hat{n}}+1\right) + \sqrt[4]{n}.\end{aligned}$$

Further, we have

$$\begin{aligned}|k(t,\tau)-k(s,\tau)| &= \left|\sqrt[3]{mt^3+\tau} - \sqrt[3]{ms^3+\tau}\right|\\ &\leqslant \sqrt[3]{|mt^3-ms^3|}\\ &= \sqrt[3]{m}\,\sqrt[3]{|t^3-s^3|}\\ &= \sqrt[3]{m}\,\sqrt[3]{|t-s|}\,\sqrt[3]{|t^2+ts+s^2|}\\ &\leqslant \sqrt[3]{3m}|t-s|^{\frac{1}{3}}\\ &= \sqrt[3]{3m}|t-s|^{\frac{1}{4}}|t-s|^{\frac{1}{12}}\\ &\leqslant \sqrt[3]{3m}|t-s|^{\frac{1}{4}}\end{aligned}$$

for all $t,s\in[0,1]$. Assumption (ii) of Theorem (3.1) is satisfied with $k_\beta = k_{\frac{1}{4}} = \sqrt[3]{3m}$. It is clear that $q(\tau) = \frac{1}{\tau+1}$ satisfies assumption (iii). The constant K is given by

$$\begin{aligned}K &= \sup\left\{\int_0^1 |k(t,\tau)|d\tau : t\in[0,1]\right\}\\ &= \sup\left\{\int_0^1 \left|\sqrt[3]{mt^3+\tau}\right| d\tau : t\in[0,1]\right\}\\ &= \int_0^1 \sqrt[3]{m+\tau}d\tau\\ &= \frac{3}{4}\left(\sqrt[3]{(m+1)^4} - \sqrt[3]{m^4}\right).\end{aligned}$$

For all $x\in H_\beta[0,1]$,

$$\begin{aligned}\|Fx\|_\beta &= |(Fx)(0)| + \sup\left\{\frac{|(Fx)(t)-(Fx)(s)|}{|t-s|^\beta} : t,s\in[0,1],\ t\neq s\right\}\\ &= \left|x^2(0)\right| + \sup\left\{\frac{\left|x^2(t)-x^2(s)\right|}{|t-s|^\beta} : t,s\in[0,1],\ t\neq s\right\}\end{aligned}$$

$$= \left|x^2(0)\right| + \sup\left\{\frac{|x(t)-x(s)|\,|x(t)+x(s)|}{|t-s|^{\beta}} : t,s\in[0,1],\ t\neq s\right\}$$

$$\leqslant \left|x^2(0)\right| + 2\|x\|_{\infty}\sup\left\{\frac{|x(t)-x(s)|}{|t-s|^{\beta}} : t,s\in[0,1],\ t\neq s\right\}$$

$$\leqslant \left|x^2(0)\right| + 2\|x\|_{\beta}\sup\left\{\frac{|x(t)-x(s)|}{|t-s|^{\beta}} : t,s\in[0,1],\ t\neq s\right\}$$

$$\leqslant \|x\|_{\beta}^2 + 2\|x\|_{\beta}^2 = 3\|x\|_{\beta}^2 .$$

Therefore, F is an operator from $H_\beta[0,1]$ into $H_\beta[0,1]$ and we can chose the function $f:\mathbb{R}^+\to\mathbb{R}^+$ as $f(x)=3x^2$. This function is non-decreasing and satisfies the inequality in assumption (iv).

Now, we will show that the operator F is continuous on the $H_\beta[0,1]$ with respect to the norm $\|.\|_\alpha$. To this end, fix arbitrarily $y\in H_\beta[0,1]$ and $\varepsilon>0$. Assume that $x\in H_\beta[0,1]$ is an arbitrary function and $\|x-y\|_\alpha<\delta$, where δ is a positive number such that $0<\delta<\sqrt{\|y\|_\alpha^2+\frac{\varepsilon}{3}}-\|y\|_\alpha$.

Then, for arbitrary $t,s\in[0,1]$ we obtain

$$\begin{aligned}
&(Fx-Fy)(t)-(Fx-Fy)(s)\\
&= x^2(t)-y^2(t)-\left(x^2(s)-y^2(s)\right)\\
&= (x(t)-y(t))(x(t)+y(t))-(x(s)-y(s))(x(s)+y(s))\\
&= [x(t)-y(t)-(x(s)-y(s))](x(t)+y(t))+(x(s)-y(s))(x(t)+y(t))\\
&-(x(s)-y(s))(x(s)+y(s))\\
&= [x(t)-y(t)-(x(s)-y(s))](x(t)+y(t))\\
&+(x(s)-y(s))[x(t)+y(t)-(x(s)+y(s))] .
\end{aligned}\tag{4.2}$$

By (4.2), we have

$$\begin{aligned}
&|(Fx-Fy)(t)-(Fx-Fy)(s)|\\
&\leqslant |x(t)-y(t)-(x(s)-y(s))|\,|x(t)+y(t)|+|x(s)-y(s)|\,|x(t)+y(t)-(x(s)+y(s))|\\
&\leqslant \|x+y\|_\infty|x(t)-y(t)-(x(s)-y(s))|+\|x-y\|_\infty|x(t)+y(t)-(x(s)+y(s))|\\
&\leqslant \|x+y\|_\alpha|x(t)-y(t)-(x(s)-y(s))|+\|x-y\|_\alpha|x(t)+y(t)-(x(s)+y(s))| .
\end{aligned}\tag{4.3}$$

By (4.3), we observe

$$
\begin{aligned}
&\sup\left\{\frac{\left|(Fx-Fy)(t)-(Fx-Fy)(s)\right|}{|t-s|^{\alpha}}:t,s\in[0,1]\,,\ t\neq s\right\}\\
&\leqslant\|x+y\|_{\alpha}\sup_{t,s\in[0,1],\ t\neq s}\frac{|x(t)-y(t)-(x(s)-y(s))|}{|t-s|^{\alpha}}\\
&+\|x-y\|_{\alpha}\sup_{t,s\in[0,1],\ t\neq s}\frac{|x(t)+y(t)-(x(s)+y(s))|}{|t-s|^{\alpha}}\\
&\leqslant\|x+y\|_{\alpha}\|x-y\|_{\alpha}+\|x-y\|_{\alpha}\|x+y\|_{\alpha}\\
&=2\|x+y\|_{\alpha}\|x-y\|_{\alpha}\,. \qquad (4.4)
\end{aligned}
$$

From (4.4), it follows

$$
\begin{aligned}
\|Fx-Fy\|_{\alpha}&=|(Fx-Fy)(0)|+\sup_{t\neq s}\left\{\frac{|(Fx-Fy)(t)-(Fx-Fy)(s)|}{|t-s|^{\alpha}}:t,s\in[0,1]\right\}\\
&\leqslant\left|x^{2}(0)-y^{2}(0)\right|+2\|x+y\|_{\alpha}\|x-y\|_{\alpha}\\
&=|x(0)-y(0)|\,|x(0)+y(0)|+2\|x+y\|_{\alpha}\|x-y\|_{\alpha}\\
&\leqslant\|x-y\|_{\infty}\|x+y\|_{\infty}+2\|x+y\|_{\alpha}\|x-y\|_{\alpha}\\
&\leqslant3\|x+y\|_{\alpha}\|x-y\|_{\alpha}\\
&\leqslant3\|x-y\|_{\alpha}\left(\|x-y\|_{\alpha}+2\|y\|_{\alpha}\right)\\
&\leqslant3\delta\left(\delta+2\|y\|_{\alpha}\right)\\
&<\varepsilon. \qquad (4.5)
\end{aligned}
$$

So that, the inequality

$$
\|Fx-Fy\|_{\alpha}\leqslant3\delta\left(\delta+2\|y\|_{\alpha}\right)<\varepsilon
$$

is satisfied for all $x\in H_{\beta}[0,1]$, where $0<\delta<\sqrt{\|y\|_{\alpha}^{2}+\varepsilon}-\|y\|_{\alpha}$. Therefore, we can chose the positive number δ as $\delta=\frac{1}{2}\sqrt{\|y\|_{\alpha}^{2}+\varepsilon}-\|y\|_{\alpha}$. This shows that the operator F is continuous at the point $y\in H_{\beta}[0,1]$. Since y is chosen arbitrarily, we deduce that F is continuous on $H_{\beta}[0,1]$ with respect to the norm $\|.\|_{\alpha}$.

In this case, the inequality appearing in assumption (v) of Theorem (3.1) takes the following form

$$
\|p\|_{\frac{1}{4}}+(2K+k_{\frac{1}{4}})rf(r)\leqslant r
$$

which is equivalent to

$$
\ln\left(\sqrt[4]{\hat{n}}+1\right)+\sqrt[4]{n}+\left[\frac{3}{2}\left(\sqrt[3]{(m+1)^{4}}-\sqrt[3]{m^{4}}\right)+\sqrt[3]{3m}\right]3r^{3}\leqslant r. \qquad (4.6)
$$

Obviously, there exists a positive number r_0 satisfying (4.6) provided that the constants $n, \hat{n}$ and m can chosen as suitable. For example, if one chose $n = \frac{1}{2^{16}}, \hat{n} = 0$ and $m = 1, r_0 = \frac{1}{4}$, then the inequality

$$\begin{aligned}
&\|p\|_{\frac{1}{4}} + (2K + k_{\frac{1}{4}})rf(r) \\
&= \ln\left(\sqrt[4]{\hat{n}} + 1\right) + \sqrt[4]{n} + \left[\frac{3}{2}\left(\sqrt[3]{(m+1)^4} - \sqrt[3]{m^4}\right) + \sqrt[3]{3m}\right] 3r^3 \\
&\approx 0,23696 < \frac{1}{4}.
\end{aligned}$$

Finally, applying Theorem (3.1) we conclude that equation (4.1) has at least one solution in the space $H_\alpha[0,1]$ with $0 < \alpha < \frac{1}{4}$.

Example 2. Let us consider the quadratic integral equation

$$x(t) = \ln\left(\frac{t}{7} + 1\right) + (ax(t) + b)\int_0^1 \sqrt{mt^2 + \tau}\, x(e^\tau)\, d\tau, \quad t \in [0,1]. \tag{4.7}$$

Set $p(t) = \ln\left(\frac{t}{7} + 1\right)$, $k(t,\tau) = \sqrt{mt^2 + \tau}$, $q(\tau) = e^\tau$ for $t, \tau \in [0,1]$ and m are non-negative constant. The operator F defined by $(Fx)(t) = ax(t) + b$, where a and b are any real number.

In what follows, we will prove that assumption (i)-(v) of Threom (3.1) are satisfied. Since function $p : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $p(t) = \ln\left(\frac{t}{7} + 1\right)$, is concav and $p(0) = 0$, then from Lemma 4.4 in [9] these functions are subadditive. By the result of subadditive

$$\begin{aligned}
|p(t) - p(s)| &= \left|\ln\left(\frac{t}{7} + 1\right) - \ln\left(\frac{s}{7} + 1\right)\right| \\
&\leqslant \ln\left|\frac{t-s}{7}\right| \\
&< \frac{|t-s|}{7} \\
&\leqslant \frac{1}{7}|t-s|^{\frac{1}{2}}
\end{aligned}$$

where we have used that $\ln x < x$ for $x > 0$. This says that $p \in H_{\frac{1}{2}}[0,1]$ (*i.e.* $\beta = \frac{1}{2}$) and, moreover, $H_p^{\frac{1}{2}} = \frac{1}{7}$. Therefore, assumption (i) of Theorem (3.1) is satisfied. Note that

$$\begin{aligned}
\|p\|_{\frac{1}{2}} &= |p(0)| + \sup\left\{\frac{|p(t) - p(s)|}{|t-s|^{\frac{1}{2}}} : t, s \in [0,1], t \neq s\right\} \\
&= |p(0)| + H_p^{\frac{1}{2}} = H_p^{\frac{1}{2}} = \frac{1}{7}.
\end{aligned}$$

Further, we have

$$\begin{aligned}|k(t,\tau)-k(s,\tau)| &= \left|\sqrt{mt^2+\tau}-\sqrt{ms^2+\tau}\right| \\ &\leqslant \sqrt{|mt^2-ms^2|} \\ &= \sqrt{m}\sqrt{|t^2-s^2|} \\ &= \sqrt{m}\sqrt{|t-s|}\sqrt{|t+s|} \\ &\leqslant \sqrt{m}\sqrt{2}|t-s|^{\frac{1}{2}} \\ &\leqslant \sqrt{2m}|t-s|^{\frac{1}{2}}\end{aligned}$$

for all $t,s\in[0,1]$. Assumption (ii) of Theorem (3.1) is satisfied with $k_\beta = k_{\frac{1}{2}} = \sqrt{2m}$. It is clear that $q(\tau)=e^\tau$ satisfies assumption (iii). In our case, the constant K is given by

$$\begin{aligned}K &= \sup\left\{\int_0^1 |k(t,\tau)|d\tau : t\in[0,1]\right\} \\ &= \sup\left\{\int_0^1 \left|\sqrt{mt^2+\tau}\right| d\tau : t\in[0,1]\right\} \\ &= \int_0^1 \sqrt{m+\tau}d\tau \\ &= \frac{2}{3}\left(\sqrt{(m+1)^3}-\sqrt{m^3}\right).\end{aligned}$$

For all $x\in H_\beta[0,1]$

$$\begin{aligned}\|Fx\|_\beta &= |(Fx)(0)| + \sup\left\{\frac{|(Fx)(t)-(Fx)(s)|}{|t-s|^\beta} : t,s\in[0,1],\ t\neq s\right\} \\ &= |ax(0)+b| + \sup\left\{\frac{|ax(t)+b-ax(s)-b|}{|t-s|^\beta} : t,s\in[0,1],\ t\neq s\right\} \\ &= |a|\,|x(0)| + |b| + \sup\left\{\frac{|x(t)-x(s)|\,|a|}{|t-s|^\beta} : t,s\in[0,1],\ t\neq s\right\} \\ &\leqslant |a|\,|x(0)| + |b| + |a|\sup\left\{\frac{|x(t)-x(s)|}{|t-s|^\beta} : t,s\in[0,1],\ t\neq s\right\} \\ &\leqslant |a|\left(|x(0)| + \sup\left\{\frac{|x(t)-x(s)|}{|t-s|^\beta} : t,s\in[0,1],\ t\neq s\right\}\right) + |b| \\ &\leqslant |a|\,\|x\|_\beta + |b|.\end{aligned}$$

Therefore, F is an operator from $H_\beta[0,1]$ into $H_\beta[0,1]$ and we can chose the function $f:\mathbb{R}^+\to\mathbb{R}^+$ as $f(x)=|a|\,x+|b|$. This function is non-decreasing and satisfies the inequality in Assumption (iv).

Now, we will show that the operator F is continuous on the $H_\beta[0,1]$ with respect to the norm $\|.\|_\alpha$. To this end, fix arbitrarily $y \in H_\beta[0,1]$ and $\varepsilon > 0$. Assume that $x \in H_\beta[0,1]$ is an arbitrary function and $\|x-y\|_\alpha < \delta$, where δ is a positive number such that $0 < \delta < \frac{\varepsilon}{|a|}$ (in this place $a \neq 0$. It is obvious that if a is zero, the operator F is continuous).

Then, for arbitrary $t, s \in [0,1]$ we obtain

$$\begin{aligned}
\|Fx - Fy\|_\alpha &= |(Fx-Fy)(0)| + \sup_{t\neq s}\left\{\frac{|(Fx-Fy)(t) - (Fx-Fy)(s)|}{|t-s|^\alpha}\right\} \\
&= |ax(0) - ay(0)| + \sup_{t\neq s}\left\{\frac{|(ax(t)-ay(t)) - (ax(s)-ay(s))|}{|t-s|^\alpha}\right\} \\
&= |a|\,|x(0)-y(0)| + |a|\sup_{t\neq s}\left\{\frac{|(x(t)-y(t)) - (x(s)-y(s))|}{|t-s|^\alpha}\right\} \\
&= |a|\left(|x(0)-y(0)| + \sup_{t\neq s}\left\{\frac{|(x(t)-y(t)) - (x(s)-y(s))|}{|t-s|^\alpha}\right\}\right) \\
&= |a|\,\|x-y\|_\alpha \\
&\leqslant |a|\,\delta \\
&< \varepsilon.
\end{aligned}$$

This shows that the operator F is continuous at the point $y \in H_\beta[0,1]$. Since y was chosen arbitrarily, we deduce that F is continuous on $H_\beta[0,1]$ with respect to the norm $\|.\|_\alpha$.

In this case, the inequality appearing in assumption (v) of Theorem (3.1) takes the following form

$$\|p\|_{\frac{1}{2}} + (2K + k_{\frac{1}{2}})rf(r) \leqslant r$$

which is equivalent to

$$\frac{1}{7} + \left[\frac{4}{3}\left(\sqrt{(m+1)^3} - \sqrt{m^3}\right) + \sqrt{2m}\right] r\,(|a|\,r + |b|) \leqslant r. \tag{4.8}$$

Obviously, there exists a number positive r_0 satisfying (4.8) provided that the constants a, b and m can chosen as suitable. For example, if one chose $a = \frac{1}{10}, b = \frac{1}{60}$ and $m = \frac{1}{2}, r_0 = \frac{1}{6}$, then the inequality

$$\begin{aligned}
&\|p\|_{\frac{1}{2}} + (2K + k_{\frac{1}{2}})r_0 f(r_0) \\
&= \frac{1}{7} + \left[\frac{4}{3}\left(\sqrt{(m+1)^3} - \sqrt{m^3}\right) + \sqrt{2m}\right] r_0\,(|a|\,r_0 + |b|) \\
&\approx 0,15939 < \frac{1}{6}.
\end{aligned}$$

Therefore, using Theorem (3.1), we conclude that equation (4.7) has at least one solution in the space $H_\alpha[0,1]$ with $0 < \alpha < \frac{1}{2} = \beta$.

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