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# Some inequalities for the polar derivative of a polynomial with restricted zeros

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Abstract: Let $p(z)$ be a polynomial of degree $n$ and for any complex number $\alpha$, $D_\alpha p(z) = np(z)+(\alpha-z)p'(z)$ denote the polar derivative of the polynomial $p(z)$ with respect to $\alpha$. In this paper, we obtain new results concerning maximum modulus of the polar derivative of a polynomial with restricted zeros. Our result generalize certain well-known polynomial inequalities.

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## 1 Introduction and statement of results

Let $p(z)$ be a polynomial of degree $n$, then according to Bernstein's inequality on the derivative of a polynomial, we have

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{1.1}$$

equality holds in (1.1) if $p(z)$ has all its zeros at the origin.
The inequality (1.1) can be sharpened, if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, in fact, P. Erdös conjectured and later Lax [9] proved that if $p(z) \neq 0$ in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

The result is best possible and equality holds in (1.2) for a polynomial which has all its zeros on $|z| = 1$.
If the polynomial $p(z)$ has all its zeros in $|z| \leq 1$, then it was proved by Turan [12] that

$$\max_{|z|=1} |p'(z)| \geq \frac{n}{2} \max_{|z|=1} |p(z)|, \tag{1.3}$$

with equality for those polynomials, which have all their zeros on $|z|=1$.
For a polynomial $p(z)$ of degree at most $n$ which having no zeros in $|z|<k,\ k\geq 1$, inequality (1.2) was generalized by Malik [10] who proved that

$$\max_{|z|=1}|p'(z)|\leq \frac{n}{1+k}\max_{|z|=1}|p(z)|. \tag{1.4}$$

The inequality (1.4) is sharp and equality holds for $p(z)=(z+k)^n$.
If the polynomial $p(z)$ has all its zeros in $|z|\leq k, k\geq 1$, then it was proved by Govil[7] that

$$\max_{|z|=1}|p'(z)|\geq \frac{n}{1+k^n}\max_{|z|=1}|p(z)|. \tag{1.5}$$

The result is best possible and equality holds in (1.5) for $p(z)=z^n+k^n$.
Let $\alpha$ be a complex number. For a polynomial $p(z)$ of degree $n$, $D_\alpha p(z)$, the polar derivative of $p(z)$ is defined as

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z).$$

It is easy to see that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$, also $D_\alpha p(z)$ generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}[\frac{D_\alpha p(z)}{\alpha}]=p'(z). \tag{1.6}$$

In order to extend inequality (1.5) for the polar derivative, Aziz and Rather[1] proved that if $p(z)$ is a polynomial of degree $n$ having all its zeros in $|z|\leq k$ where $k\geq 1$, then for every real or complex number $\alpha$ with $|\alpha|\geq k$,

$$\max_{|z|=1}|D_\alpha p(z)|\geq n\frac{|\alpha|-k}{1+k^n}\max_{|z|=1}|p(z)|. \tag{1.7}$$

The bounds are obtained depends only on the zero of largest modulus and not on the other zeros even if some of them are close to the origin. Therefore, it would be interesting to obtain a bound, which depends on the location of all the zeros of a polynomial. In this connection we use some known ideas in the literature and obtain the following interesting results.

**Theorem 1.1** *Let*

$$p(z)=\sum_{\nu=0}^{n}a_\nu z^\nu=a_n\prod_{\nu=1}^{n}(z-z_\nu),\ a_n\neq 0,$$

*be a polynomial of degree $n$, $|z_\nu|\leq k_\nu,\ 1\leq\nu\leq n$, and $k=\max(k_1,k_2,...,k_n)\geq 1$.*

*Then for every real or complex number $\alpha$ with $|\alpha| \geq k$, we have*

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \geq \frac{|\alpha|-k}{1+k^n}\sum_{\nu=1}^{n}\frac{k}{k+k_\nu}[2\max_{|z|=1}|p(z)| + \frac{k^n-1}{k^n}\min_{|z|=k}|p(z)| \\ +\frac{4|a_{n-1}|}{k(n+1)}(\frac{k^n-1-n(k-1)}{n}) \\ +\frac{4|a_{n-2}|}{k^2}((\frac{(k^n-1)-n(k-1)}{n(n-1)}) - (\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}))] \\ +\frac{2(k^{n-1}-1)}{(n+1)k^{n-1}}|na_0+\alpha a_1| + \frac{1}{k^{n-1}}[\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}]|(n-1)a_1+2\alpha a_2|,\end{aligned} \tag{1.8}$$

*for $n > 3$*
*and*

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \geq \frac{|\alpha|-k}{1+k^n}\sum_{\nu=1}^{n}\frac{k}{k+k_\nu}[2\max_{|z|=1}|p(z)| + \frac{k^n-1}{k^n}\min_{|z|=k}|p(z)| \\ + \frac{4|a_{n-1}|}{k(n+1)}(\frac{(k^n-1)-n(k-1)}{n}) + \frac{4|a_{n-2}|(k-1)^n}{k^2n(n-1)}] \\ + \frac{(k^2-1)}{2k^{n-1}}|na_0+\alpha a_1| + \frac{(k-1)^2}{2k^{n-1}}|(n-1)a_1+2\alpha a_2|,\end{aligned} \tag{1.9}$$

*for $n = 3$.*

Since $\dfrac{k}{k+k_\nu} \geq \dfrac{1}{2}$ for $1 \leq \nu \leq n$, the above theorem gives the following result which is an improvement of the inequality (1.7).

**Corollary 1.2** *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree $n$ having all its zeros in $|z| \leq k,\ k \geq 1$, then for every real or complex number $\alpha$ with $|\alpha| \geq k$, we have*

$$\begin{aligned}&\max_{|z|=1}|D_\alpha p(z)| \geq \frac{n(|\alpha|-k)}{1+k^n}[\max_{|z|=1}|p(z)| + \frac{k^n-1}{2k^n}\min_{|z|=k}|p(z)| \\ &+ \frac{2|a_{n-1}|}{(n+1)k}(\frac{k^n-1-n(k-1)}{n}) \\ &+ \frac{2|a_{n-2}|}{k^2}((\frac{(k^n-1)-n(k-1)}{n(n-1)}) - (\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}))] \\ &+ \frac{2(k^{n-1}-1)}{(n+1)k^{n-1}}|na_0+\alpha a_1| + \frac{1}{k^{n-1}}[\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}]|(n-1)a_1+2\alpha a_2|,\end{aligned} \tag{1.10}$$

*for $n > 3$*

*and*

$$\max_{|z|=1}|D_\alpha p(z)| \geq \frac{n(|\alpha|-k)}{1+k^n}[\max_{|z|=1}|p(z)| + \frac{k^n-1}{2k^n}\min_{|z|=k}|p(z)| \\ + \frac{2|a_{n-1}|}{(n+1)k}(\frac{(k^n-1)-n(k-1)}{n}) + \frac{2|a_{n-2}|(k-1)^n}{k^2n(n-1)}] \\ + \frac{(k^2-1)}{2k^{n-1}}|na_0+\alpha a_1| + \frac{(k-1)^2}{2k^{n-1}}|(n-1)a_1+2\alpha a_2|, \tag{1.11}$$

*for* $n=3$.

Dividing both sides of the inequalities (1.10) and (1.11) by $|\alpha|$ and letting $|\alpha| \to \infty$, we have the following refinement of the inequality (1.5).

**Corollary 1.3** *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* $n$*, having all its zeros in* $|z| \leq k$, $k \geq 1$*, then for every real or complex number* $\alpha$ *with* $|\alpha| \geq k$*, we have*

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{1+k^n}[\max_{|z|=1}|p(z)| + \frac{k^n-1}{2k^n}\min_{|z|=k}|p(z)| \\ + \frac{2|a_{n-1}|}{(n+1)k}(\frac{k^n-1-n(k-1)}{n}) \\ + \frac{2|a_{n-2}|}{k^2}((\frac{(k^n-1)-n(k-1)}{n(n-1)}) - (\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}))] \\ + \frac{2(k^{n-1}-1)}{(n+1)k^{n-1}}|a_1| + \frac{1}{k^{n-1}}[\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}]|2a_2|, \tag{1.12}$$

*for* $n > 3$
*and*

$$\max_{|z|=1}|p'(z)| \geq \frac{n}{1+k^n}[\max_{|z|=1}|p(z)| + \frac{k^n-1}{2k^n}\min_{|z|=k}|p(z)| \\ + \frac{2|a_{n-1}|}{(n+1)k}(\frac{(k^n-1)-n(k-1)}{n}) + \frac{2|a_{n-2}|(k-1)^n}{k^2n(n-1)}] \\ + \frac{(k^2-1)}{2k^{n-1}}|a_1| + \frac{(k-1)^2}{2k^{n-1}}|2a_2|, \tag{1.13}$$

*for* $n=3$.

As an application of Theorem 1.1 we prove the following result.

**Theorem 1.4** *Let*

$$p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu = a_n \prod_{\nu=1}^{n}(z-z_\nu), \ a_n \neq 0,$$

*be a polynomial of degree $n$, $|z_\nu| \geq k_\nu$, $1 \leq \nu \leq n$, and $k = \min(k_1, k_2, ..., k_n) \leq 1$. Then for every real or complex number $\delta$ with $|\delta| \leq k$, we have*

$$\max_{|z|=1}|D_\delta p(z)| \geq \frac{(k-|\delta|)k^{n-1}}{1+k^n}\sum_{\nu=1}^{n}\frac{k_\nu}{k+k_\nu}[2\max_{|z|=1}|p(z)| + \frac{1-k^n}{k^n}\min_{|z|=k}|p(z)|$$
$$+\frac{4|a_1|k}{(n+1)}(\frac{(1-k^n)-n(k^{n-1}-k^n)}{nk^n}) + 4|a_2|k^2((\frac{(1-k^n)-n(k^{n-1}-k^n)}{n(n-1)k^n})-$$
$$(\frac{(1-k^{n-2})-(n-2)(k^{n-3}-k^{n-2})}{(n-2)(n-3)k^{n-2}}))] \tag{1.14}$$
$$+\frac{2(1-k^{n-1})}{(n+1)}|na_n+\alpha a_{n-1}|+$$
$$k^{n-1}[\frac{(1-k^{n-1})}{(n-1)k^{n-1}} - \frac{(1-k^{n-3})}{(n-3)k^{n-3}}]|(n-1)a_{n-1}+2\alpha a_{n-2}|,$$

*for $n > 3$*
*and*

$$\max_{|z|=1}|D_\delta p(z)| \geq \frac{(k-|\delta|)k^{n-1}}{1+k^n}\sum_{\nu=1}^{n}\frac{k_\nu}{k+k_\nu}[2\max_{|z|=1}|p(z)| + \frac{1-k^n}{k^n}\min_{|z|=k}|p(z)|$$
$$+\frac{4|a_1|k}{(n+1)}(\frac{(1-k^n)-n(k^{n-1}-k^n)}{nk^n}) + 4|a_2|k^2(1-k)^n] \tag{1.15}$$
$$+\frac{1-k^2}{2}|na_n+\alpha a_{n-1}| + \frac{(1-k)^2}{2}|(n-1)a_{n-1}+2\alpha a_{n-2}|,$$

*for $n = 3$.*

Since $\dfrac{k_\nu}{k+k_\nu} \geq \dfrac{1}{2}$ for $1 \leq \nu \leq n$, then Theorem 1.4 gives the following result.

**Corollary 1.5** *Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree $n$ does not vanish in $|z < k$ where $k \leq 1$. Then for every real or complex number $\delta$ with $|\delta| \leq k$, we have*

$$\max_{|z|=1}|D_\delta p(z)| \geq \frac{n(k-|\delta|)k^{n-1}}{1+k^n}[\max_{|z|=1}|p(z)| + \frac{1-k^n}{2k^n}\min_{|z|=k}|p(z)|$$
$$+\frac{2|a_1|k}{(n+1)}(\frac{(1-k^n)-n(k^{n-1}-k^n)}{nk^n}) + 2|a_2|k^2((\frac{(1-k^n)-n(k^{n-1}-k^n)}{n(n-1)k^n})-$$
$$(\frac{(1-k^{n-2})-(n-2)(k^{n-3}-k^{n-2})}{(n-2)(n-3)k^{n-2}}))] \tag{1.16}$$
$$+\frac{2(1-k^{n-1})}{(n+1)}|na_n+\alpha a_{n-1}|+$$
$$k^{n-1}[\frac{(1-k^{n-1})}{(n-1)k^{n-1}} - \frac{(1-k^{n-3})}{(n-3)k^{n-3}}]|(n-1)a_{n-1}+2\alpha a_{n-2}|,$$

*for* $n > 3$
*and*

$$\max_{|z|=1}|D_\delta p(z)| \geq \frac{n(k-|\delta|)k^{n-1}}{1+k^n}[\max_{|z|=1}|p(z)| + \frac{1-k^n}{2k^n}\min_{|z|=k}|p(z)|$$
$$+\frac{2|a_1|k}{(n+1)}(\frac{(1-k^n)-n(k^{n-1}-k^n)}{nk^n}) + 2|a_2|k^2(1-k)^n] \tag{1.17}$$
$$+\frac{1-k^2}{2}|na_n+\alpha a_{n-1}| + \frac{(1-k)^2}{2}|(n-1)a_{n-1}+2\alpha a_{n-2}|,$$

*for* $n = 3$.

## 2 Lemmas

For proof of the theorems, we need the following lemmas. The first lemma is due to Dewan, Kaur and Mir [4].

**Lemma 2.1** *If* $p(z)$ *is a polynomial of degree* $n$, *then for* $R \geq 1$,

$$\begin{aligned}\max_{|z|=R}|p(z)| \leq R^n\max_{|z|=1}|p(z)| - \frac{2(R^n-1)}{n+2}|p(0)|\\ -[\frac{R^n-1}{n}-\frac{R^{n-2}-1}{n-2}]|p'(0)|\end{aligned} \tag{2.1}$$

*if* $n > 2$, *and*

$$\max_{|z|=R}|p(z)| \leq R^n\max_{|z|=1}|p(z)| - \frac{R-1}{2}[(R+1)|p(0)| + (R-1)|p'(0)|] \tag{2.2}$$

*if* $n = 2$.

**Lemma 2.2** *If* $p(z)$ *is a polynomial of degree* $n$, *having no zeros in* $|z| < 1$, *then for* $R \geq 1$ ,

$$\begin{aligned}\max_{|z|=R}|p(z)| \leq \frac{R^n+1}{2}\max_{|z|=1}|p(z)| - \frac{R^n-1}{2}\min_{|z|=1}|p(z)| -\\ \frac{2}{n+1}[\frac{(R^n-1)}{n} - (R-1)]|p'(0)| -\\ [\frac{(R^n-1)-n(R-1)}{n(n-1)} - \frac{(R^{n-2}-1)-(n-2)(R-1)}{(n-2)(n-3)}]|p''(0)|\end{aligned} \tag{2.3}$$

*if* $n > 3$, *and*

$$\begin{aligned}\max_{|z|=R}|p(z)| \leq \frac{R^n+1}{2}\max_{|z|=1}|p(z)| - \frac{R^n-1}{2}\min_{|z|=1}|p(z)|\\ -\frac{2}{n+1}[\frac{R^n-1}{n} - (R-1)]|p'(0)|\\ -\frac{(R-1)^n}{n(n-1)}|p''(0)|]\end{aligned} \tag{2.4}$$

*if* $n = 3$.

This lemma is due to Dewan, Singh and Mir [5].

**Lemma 2.3** *If* $p(z) = a_n \prod_{\nu=1}^{n}(z - z_\nu),\ a_n \neq 0$, *is a polynomial of degree* $n$, *such that* $|z_\nu| \leq 1,\ 1 \leq \nu \leq n$, *then*

$$\max_{|z|=1} |p'(z)| \geq \sum_{\nu=1}^{n} \frac{1}{1 + |z_\nu|} \max_{|z|=1} |p(z)|. \tag{2.5}$$

This lemma is due to Giroux, Rahman and Schmeisser [8].

**Lemma 2.4** *If* $p(z)$ *is a polynomial of degree* $n$ *and* $\alpha$ *is any real or complex number with* $|\alpha| \neq 0$, *then for* $|z| = 1$

$$|D_\alpha q(z)| = |n\overline{\alpha}p(z) + (1 - \overline{\alpha}z)p'(z)|, \tag{2.6}$$

*where* $q(z) = z^n \overline{p(\frac{1}{\overline{z}})}$.

This lemma is due to Aziz [2].

# 3 Proofs of the theorems

**Proof of the Theorem 1.1.** Let $G(z) = p(kz)$. Since all the zeros of $p(z)$ lie in $|z| \leq k$, then all the zeros of $G(z)$ lie in $|z| \leq 1$. Now on applying Lemma 2.3 to the polynomial $G(z)$, we get

$$\max_{|z|=1} |G'(z)| \geq \sum_{\nu=1}^{n} \frac{1}{1 + \frac{|z_\nu|}{k}} \max_{|z|=1} |G(z)|. \tag{3.1}$$

Let $H(z) = z^n \overline{G(1/\overline{z})}$. Then it can be easily verified that for $|z| = 1$

$$|H'(z)| = |nG(z) - zG'(z)|. \tag{3.2}$$

The polynomial $G(z)$ has all its zeros in $|z| \leq 1$ and $|H(z)| = |G(z)|$ for $|z| = 1$, therefore, by Gauss-Lucas theorem for $|z| = 1$, we have

$$|H'(z)| \leq |G'(z)|. \tag{3.3}$$

Now for every real or complex number $\alpha$ with $|\alpha| \geq k$, we have

$$|D_{\frac{\alpha}{k}} G(z)| = |nG(z) - zG'(z) + \frac{\alpha}{k} G'(z)| \geq |\frac{\alpha}{k}||G'(z)| - |nG(z) - zG'(z)|. \tag{3.4}$$

This gives with the help of (3.2) and (3.3) that

$$\max_{|z|=1} |D_{\frac{\alpha}{k}} G(z)| \geq \frac{|\alpha| - k}{k} \max_{|z|=1} |G'(z)|. \tag{3.5}$$

Using (3.1) in (3.4), we get

$$\max_{|z|=1}|D_{\frac{\alpha}{k}}G(z)| \geq \frac{|\alpha|-k}{k}\sum_{\nu=1}^{n}\frac{k}{k+|z_\nu|}\max_{|z|=1}|G(z)|. \tag{3.6}$$

Replacing $G(z)$ by $p(kz)$, we get

$$\max_{|z|=1}|D_{\frac{\alpha}{k}}p(kz)| \geq (|\alpha|-k)\sum_{\nu=1}^{n}\frac{1}{k+|z_\nu|}\max_{|z|=1}|p(kz)|. \tag{3.7}$$

which implies

$$\max_{|z|=1}|np(kz)+(\frac{\alpha}{k}-z)kp'(kz)| \geq (|\alpha|-k)\sum_{\nu=1}^{n}\frac{1}{k+|z_\nu|}\max_{|z|=1}|p(kz)|.$$

which gives

$$\max_{|z|=k}|D_\alpha p(z)| \geq (|\alpha|-k)\sum_{\nu=1}^{n}\frac{1}{k+|z_\nu|}\max_{|z|=k}|p(z)|. \tag{3.8}$$

The polynomial $p(z)$ is of degree $n>3$ and so $D_\alpha p(z)$ is the polynomial of degree $n-1$, where $n-1>2$, hence applying Lemma 2.1 to the polynomial $D_\alpha p(z)$, we get

$$\begin{aligned}\max_{|z|=k}|D_\alpha p(z)| \leq k^{n-1}\max_{|z|=1}|D_\alpha p(z)| - \frac{2(k^{n-1}-1)}{n+1}|na_0+\alpha a_1|\\ -[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}]|(n-1)a_1+2\alpha a_2|.\end{aligned} \tag{3.9}$$

Combining (3.9) and (3.8), we get

$$\begin{aligned}&\max_{|z|=1}|D_\alpha p(z)| \geq \frac{|\alpha|-k}{k^{n-1}}[\sum_{\nu=1}^{n}\frac{1}{k+|z_\nu|}\max_{|z|=k}|p(z)|]+\\ &\frac{2(k^{n-1}-1)}{(n+1)k^{n-1}}|na_0+\alpha a_1| + \frac{1}{k^{n-1}}[\frac{k^{n-1}-1}{n-1}-\frac{k^{n-3}-1}{n-3}]|(n-1)a_1+2\alpha a_2|.\end{aligned} \tag{3.10}$$

Since the polynomial $p(z)$ has all zeros in $|z|\leq k,\ k\geq 1$, then $q(z)=z^np(1/z)$ has no zero in $|z|<1/k$, hence the polynomial $q(z/k)$ having no zeros in $|z|<1$, therefore on applying Lemma 2.2 to the polynomial $q(z/k)$, we get

$$\begin{aligned}&\max_{|z|=k}|q(z/k)| \leq\\ &[\frac{k^n+1}{2}]\max_{|z|=1}|q(z/k)-(\frac{k^n-1}{2})\min_{|z|=1}|q(z/k)| - \frac{2|a_{n-1}|}{(n+1)k}[\frac{k^n-1}{n}-(k-1)]\\ &-\frac{2|a_{n-2}|}{k^2}[(\frac{(k^n-1)-n(k-1)}{n(n-1)})-(\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)})].\end{aligned} \tag{3.11}$$

Since $\max\limits_{|z|=1}|q(z/k)| = (1/k^n)\max\limits_{|z|=k}|p(z)|$, and $\min\limits_{|z|=1}|q(z/k)| = (1/k^n)\min\limits_{|z|=k}|p(z)|$, then (3.11) is equivalent to

$$\begin{aligned}
&\max_{|z|=k}|p(z)| \geq (\frac{2k^n}{k^n+1})\max_{|z|=1}|p(z) + (\frac{k^n-1}{k^n+1})\min_{|z|=k}|p(z)| \\
&+ \frac{4k^{n-1}|a_{n-1}|}{(n+1)(k^n+1)}[\frac{k^n-1}{n} - (k-1)] \\
&+ \frac{4k^{n-2}|a_{n-2}|}{k^n+1}[(\frac{(k^n-1)-n(k-1)}{n(n-1)}) - (\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)})].
\end{aligned} \tag{3.12}$$

Combining (3.10) and (3.12), we get

$$\begin{aligned}
&\max_{|z|=1}|D_\alpha p(z)| \geq \frac{|\alpha|-k}{k^{n-1}}\sum_{\nu=1}^{n}\frac{1}{k+k_\nu}[(\frac{2k^n}{k^n+1})\max_{|z|=1}|p(z)| \\
&+ (\frac{k^n-1}{k^n+1})\min_{|z|=k}|p(z)| + \frac{4k^{n-1}|a_{n-1}|}{(n+1)(k^n+1)}(\frac{k^n-1}{n} - (k-1)) \\
&+ \frac{4k^{n-2}|a_{n-2}|}{k^n+1}((\frac{(k^n-1)-n(k-1)}{n(n-1)}) - (\frac{(k^{n-2}-1)-(n-2)(k-1)}{(n-2)(n-3)}))] \\
&+ \frac{2(k^{n-1}-1)}{(n+1)k^{n-1}}|na_0+\alpha a_1| + \frac{1}{k^{n-1}}[\frac{k^{n-1}-1}{n-1} - \frac{k^{n-3}-1}{n-3}] \\
&\times |(n-1)a_1 + 2\alpha a_2|.
\end{aligned} \tag{3.13}$$

which completes the proof of (1.10). The proof of the Theorem 1.1 in the case $n = 3$ follows along the same lines as the proof of (1.10) but instead of inequalities (2.1) and (2.3), we use inequalities (2.2) and (2.4), respectively. □

**Proof of the Theorem 1.4.** Let $q(z) = z^n\overline{p(1/\overline{z})}$ then $1/|z_\nu| \leq 1/k_\nu$ for $1 \leq \nu \leq n$ such that $1/k = \max(1/k_1, 1/k_2, \cdots, 1/k_n) \geq 1$. On applying Theorem 1.1 to the polynomial $q(z)$, we get

$$\begin{aligned}
&\max_{|z|=1}|D_\alpha q(z)| \geq (|\alpha| - 1/k)k^{n-1}\sum_{\nu=1}^{n}\frac{1}{1/k+1/k_\nu}[(\frac{2/k^n}{1/k^n+1})\max_{|z|=1}|q(z)| \\
&+ (\frac{1/k^n-1}{1/k^n+1})\min_{|z|=1/k}|q(z)| + \frac{4|a_1|}{(1+1/k^n)k^{n-1}(n+1)}(\frac{(1/k^n-1)-n(1/k-1)}{n}) \\
&+ \frac{4|a_2|}{k^{n-2}(1+1/k^n)}((\frac{(1/k^n-1)-n(1/k-1)}{n(n-1)}) - (\frac{(1/k^{n-2}-1)-(n-2)(1/k-1)}{(n-2)(n-3)}))] \\
&+ \frac{2k^{n-1}(1/k^{n-1}-1)}{(n+1)}|na_n + \alpha a_{n-1}| + k^{n-1}[\frac{(1/k^{n-1}-1)}{n-1} - \frac{(1/k^{n-3}-1)}{n-3}] \\
&\times |(n-1)a_{n-1} + 2\alpha a_{n-2}|
\end{aligned} \tag{3.14}$$

Now from lemma 2.4 it follows that for $|z| = 1$, $|D_\alpha q(z)| = |\alpha||D_{\frac{1}{\alpha}} p(z)|$ Using the above equality in (3.14), we get for $|\alpha| \geq 1/k$,

$$\begin{aligned}
|\alpha| \max_{|z|=1} |D_{\frac{1}{\alpha}} p(z)| \geq (|\alpha| - 1/k)k^{n-1} \sum_{\nu=1}^{n} \frac{kk_\nu}{k+k_\nu}[(\frac{2}{1+k^n}) \max_{|z|=1} |p(z)| \\
+ \frac{1}{k^n}(\frac{1-k^n}{1+k^n}) \min_{|z|=k} |p(z)| + \frac{4|a_1|k}{(n+1)(k^n+1)}(\frac{(1-k^n) - n(k^{n-1}-k^n)}{nk^n}) \\
+ \frac{4|a_2|k^2}{(1+k^n)}((\frac{(1-k^n) - n(k^{n-1}-k^n)}{n(n-1)k^n}) - (\frac{(1-k^{n-2}) - (n-2)(k^{n-3}-k^{n-2})}{(n-2)(n-3)k^{n-2}}))] \\
+ \frac{2(1-k^{n-1})}{(n+1)}|na_n + \alpha a_{n-1}| + k^{n-1}[\frac{(1-k^{n-1})}{(n-1)k^{n-1}} - \frac{(1-k^{n-3})}{(n-3)k^{n-3}}] \\
\times |(n-1)a_{n-1} + 2\alpha a_{n-2}|,
\end{aligned} \tag{3.15}$$

Replacing $\frac{1}{\alpha}$ by $\delta$, so that $|\delta| \leq k$, we get from (3.15)

$$\begin{aligned}
|\frac{1}{\delta}| \max_{|z|=1} |D_\delta p(z)| \geq (|\frac{1}{\delta}| - 1/k)k^{n-1} \sum_{\nu=1}^{n} \frac{kk_\nu}{k+k_\nu}[(\frac{2}{1+k^n}) \max_{|z|=1} |p(z)| \\
+ \frac{1}{k^n}(\frac{1-k^n}{1+k^n}) \min_{|z|=k} |p(z)| + \frac{4|a_1|k}{(n+1)(k^n+1)}(\frac{(1-k^n) - n(k^{n-1}-k^n)}{nk^n}) \\
+ \frac{4|a_2|k^2}{(1+k^n)}((\frac{(1-k^n) - n(k^{n-1}-k^n)}{n(n-1)k^n}) - (\frac{(1-k^{n-2}) - (n-2)(k^{n-3}-k^{n-2})}{(n-2)(n-3)k^{n-2}}))] \\
+ \frac{2(1-k^{n-1})}{(n+1)}|na_n + \alpha a_{n-1}| + k^{n-1}[\frac{(1-k^{n-1})}{(n-1)k^{n-1}} - \frac{(1-k^{n-3})}{(n-3)k^{n-3}}] \\
\times |(n-1)a_{n-1} + 2\alpha a_{n-2}|.
\end{aligned} \tag{3.16}$$

Or

$$\begin{aligned}
\max_{|z|=1} |D_\delta p(z)| \geq (k - |\delta|)k^{n-1} \sum_{\nu=1}^{n} \frac{k_\nu}{k+k_\nu}[(\frac{2}{1+k^n}) \max_{|z|=1} |p(z)| \\
+ \frac{1}{k^n}(\frac{1-k^n}{1+k^n}) \min_{|z|=k} |p(z)| + \frac{4|a_1|k}{(n+1)(k^n+1)}(\frac{(1-k^n) - n(k^{n-1}-k^n)}{nk^n}) \\
+ \frac{4|a_2|k^2}{(1+k^n)}((\frac{(1-k^n) - n(k^{n-1}-k^n)}{n(n-1)k^n}) - (\frac{(1-k^{n-2}) - (n-2)(k^{n-3}-k^{n-2})}{(n-2)(n-3)k^{n-2}}))] \\
+ \frac{2(1-k^{n-1})}{(n+1)}|na_n + \alpha a_{n-1}| + k^{n-1}[\frac{(1-k^{n-1})}{(n-1)k^{n-1}} - \frac{(1-k^{n-3})}{(n-3)k^{n-3}}] \\
\times |(n-1)a_{n-1} + 2\alpha a_{n-2}|.
\end{aligned} \tag{3.17}$$

Which is (1.14). The proof of the Theorem 1.4 in the case $n = 3$ follows along the same lines as the proof of Theorem 1.1. Hence the proof of Theorem 1.4 is complete. □

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