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Some New Existence Results and Stability Concepts for Fractional Partial Random Differential Equations

Saïd Abbas, Mouffak Benchohra and Mohamed Abdalla Darwish

ABSTRACT: In the present paper we provide some existence results and Ulam's type stability concepts for the Darboux problem of partial fractional random differential equations in Banach spaces, by applying the measure of noncompactness and a random fixed point theorem with stochastic domain.

AMS Subject Classification: 26A33, 34G20, 34A40, 45N05, 47H10.

Keywords and Phrases: Random differential equation; Left-sided mixed Riemann-Liouville integral; Caputo fractional order derivative; Banach space; Darboux problem; Ulam stability.

1. Introduction

The fractional calculus deals with extensions of derivatives and integrals to noninteger orders. The field of fractional differential equations has been subjected to an intensive development of the theory and the applications in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering [22, 36, 42]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [4, 5], Kilbas *et al.* [29], and Zhou [51], the papers of Abbas *et al.* [1, 2, 3, 6, 7, 8], Baleanu *et al.* [11], Darwish *et al.* [16, 17, 18], Vityuk *et al.* [44, 45, 46], and the references therein.

On the other hand, due to a combination of uncertainties and complexities, deterministic equations can hardly describe a real system precisely. In order to take random factors into account, many stochastic models were proposed and various achievements were obtained; see for instance the book by Soon [41], and the references therein.

The initial value problems of ordinary random differential equations have been studied in the literature on bounded as well as unbounded intervals of the real line for different aspects of the solution. See for example, Burton and Furumochi [14] and the references therein.

The stability of functional equations was originally raised by Ulam in 1940 in a talk given at Wisconsin University. The problem posed by Ulam was the following: Under what conditions does there exist an additive mapping near an approximately additive mapping? (for more details see [43]). The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces [23]. Thereafter, this type of stability is called the Ulam-Hyers stability. In 1978, Rassias [38] provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus, the stability question of functional equations is how do the solutions of the inequality differ from those of the given functional equation? Considerable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all kinds of functional equations; see the monographs [24, 26]. Bota-Boriceanu and Petrusel [13], Petru *et al.* [34, 35], and Rus [39, 40] discussed the Ulam-Hyers stability for operatorial equations and inclusions. Castro and Ramos [15], and Jung [28] considered the Hyers-Ulam-Rassias stability for a class of Volterra integral equations. Ulam stability for fractional differential equations with Caputo derivative are proposed by Wang *et al.* [47, 48]. Some stability results for fractional integral equation are obtained by Wei *et al.* [49]. More details from historical point of view, and recent developments of such stabilities are reported in [27, 39, 49].

In this paper, we discuss the existence of random solutions and Ulam stabilities for the following fractional partial random differential equations

$${}^c D_{\theta}^r u(x, y, w) = f(x, y, u(x, y, w), w); \text{ for a.a. } (x, y) \in J := [0, a] \times [0, b], w \in \Omega, \quad (1)$$

with the initial conditions

$$\begin{cases} u(x, 0, w) = \varphi(x, w); & x \in [0, a], \\ u(0, y, w) = \psi(y, w); & y \in [0, b], \\ \varphi(0, w) = \psi(0, w), \end{cases} \quad w \in \Omega, \quad (2)$$

where $a, b > 0$, $\theta = (0, 0)$, ${}^c D_{\theta}^r$ is the fractional Caputo derivative of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, (Ω, \mathcal{A}) is a measurable space, $f : J \times E \times \Omega \rightarrow E$ is a given continuous function, $(E, \|\cdot\|_E)$ is a real Banach space, $\varphi : [0, a] \times \Omega \rightarrow E$, $\psi : [0, b] \times \Omega \rightarrow E$ are given functions such that $\varphi(\cdot, w)$ and $\psi(\cdot, w)$ are absolutely continuous functions for all $w \in \Omega$, and $\varphi(x, \cdot)$ and $\psi(y, \cdot)$ are measurable for all $x \in [0, a]$ and $y \in [0, b]$ respectively, and \mathcal{C} is the Banach space of all continuous functions from J into E with the supremum (uniform) norm $\|\cdot\|_{\infty}$. This paper initiates the existence and Ulam stabilities of random solutions via fixed point techniques.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Denote $L^1(J)$ the space of Bochner-integrable functions $u : J \rightarrow E$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(x, y)\|_E dy dx.$$

$L^\infty(J)$ the Banach space of functions $u : J \rightarrow \mathbb{R}$ which are essentially bounded. As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into E .

Let β_E be the σ -algebra of Borel subsets of E . A mapping $v : \Omega \rightarrow E$ is said to be measurable if for any $B \in \beta_E$, one has

$$v^{-1}(B) = \{w \in \Omega : v(w) \in B\} \subset \mathcal{A}.$$

To define integrals of sample paths of random process, it is necessary to define a jointly measurable map.

Definition 2.1. A mapping $T : \Omega \times E \rightarrow E$ is called jointly measurable if for any $B \in \beta_E$, one has

$$T^{-1}(B) = \{(w, v) \in \Omega \times E : T(w, v) \in B\} \subset \mathcal{A} \times \beta_E,$$

where $\mathcal{A} \times \beta_E$ is the direct product of the σ -algebras \mathcal{A} and β_E those defined in Ω and E respectively.

Lemma 2.2. [19] Let $T : \Omega \times E \rightarrow E$ be a mapping such that $T(\cdot, v)$ is measurable for all $v \in E$, and $T(w, \cdot)$ is continuous for all $w \in \Omega$. Then the map $(w, v) \mapsto T(w, v)$ is jointly measurable.

Definition 2.3. [21] A function $f : J \times E \times \Omega \rightarrow E$ is called random Carathéodory if the following conditions are satisfied:

- (i) The map $(x, y, w) \rightarrow f(x, y, u, w)$ is jointly measurable for all $u \in E$, and
- (ii) The map $u \rightarrow f(x, y, u, w)$ is continuous for almost all $(x, y) \in J$ and $w \in \Omega$.

Let $T : \Omega \times E \rightarrow E$ be a mapping. Then T is called a random operator if $T(w, u)$ is measurable in w for all $u \in E$ and it is expressed as $T(w)u = T(w, u)$. In this case we also say that $T(w)$ is a random operator on E . A random operator $T(w)$ on E is called continuous (resp. compact, totally bounded and completely continuous) if $T(w, u)$ is continuous (resp. compact, totally bounded and completely continuous) in u for all $w \in \Omega$. The details of completely continuous random operators in Banach spaces and their properties appear in Itoh [25].

Definition 2.4. [20] Let $\mathcal{P}(Y)$ be the family of all nonempty subsets of Y and C be a mapping from Ω into $\mathcal{P}(Y)$. A mapping $T : \{(w, y) : w \in \Omega, y \in C(w)\} \rightarrow Y$ is called random operator with stochastic domain C if C is measurable (i.e., for all closed $A \subset Y$, $\{w \in \Omega, C(w) \cap A \neq \emptyset\}$ is measurable) and for all open $D \subset Y$ and all $y \in Y$, $\{w \in \Omega : y \in C(w), T(w, y) \in D\}$ is measurable. T will be called continuous if every $T(w)$ is continuous. For a random operator T , a mapping $y : \Omega \rightarrow Y$ is called random (stochastic) fixed point of T if for P -almost all $w \in \Omega$, $y(w) \in C(w)$ and $T(w)y(w) = y(w)$ and for all open $D \subset Y$, $\{w \in \Omega : y(w) \in D\}$ is measurable.

Let \mathcal{M}_X denote the class of all bounded subsets of a metric space X .

Definition 2.5. Let X be a complete metric space. A map $\alpha : \mathcal{M}_X \rightarrow [0, \infty)$ is called a measure of noncompactness on X if it satisfies the following properties for all $B, B_1, B_2 \in \mathcal{M}_X$.

(MNC.1) $\alpha(B) = 0$ if and only if B is precompact (Regularity),

(MNC.2) $\alpha(B) = \alpha(\overline{B})$ (Invariance under closure),

(MNC.3) $\alpha(B_1 \cup B_2) = \alpha(B_1) + \alpha(B_2)$ (Semi-additivity).

For more details on measure of noncompactness and its properties see [9].

Example 2.6. In every metric space X , the map $\phi : \mathcal{M}_X \rightarrow [0, \infty)$ with $\phi(B) = 0$ if B is relatively compact and $\phi(B) = 1$ otherwise is a measure of noncompactness, the so-called discrete measure of noncompactness [[10], Example1, p. 19].

Let $\theta = (0, 0)$, $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $f \in L^1(J)$, the expression

$$(I_\theta^r f)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t) dt ds,$$

is called the left-sided mixed Riemann-Liouville integral of order r , where $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt$; $\xi > 0$.

In particular,

$$(I_\theta^\sigma u)(x, y) = u(x, y), \quad (I_\theta^\sigma u)(x, y) = \int_0^x \int_0^y u(s, t) dt ds; \text{ for almost all } (x, y) \in J,$$

where $\sigma = (1, 1)$.

For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0, \infty)$, when $u \in L^1(J)$. Note also that when $u \in \mathcal{C}$, then $(I_\theta^r u) \in \mathcal{C}$, moreover

$$(I_\theta^r u)(x, 0) = (I_\theta^r u)(0, y) = 0; \quad x \in [0, a], \quad y \in [0, b].$$

By $1-r$ we mean $(1-r_1, 1-r_2) \in [0, 1) \times [0, 1)$. Denote by $D_{xy}^2 := \frac{\partial^2}{\partial x \partial y}$, the mixed second order partial derivative.

Definition 2.7. [4, 46] Let $r \in (0, 1) \times (0, 1)$ and $u \in L^1(J)$. The Caputo fractional-order derivative of order r of u is defined by the expression

$${}^c D_{\theta}^r u(x, y) = (I_{\theta}^{1-r} D_{xy}^2 u)(x, y).$$

The case $\sigma = (1, 1)$ is included and we have

$$({}^c D_{\theta}^{\sigma} u)(x, y) = (D_{xy}^2 u)(x, y); \text{ for almost all } (x, y) \in J.$$

Definition 2.8. By a random solution of the random problem (1)-(2) we mean a measurable function $u : \Omega \rightarrow AC(J)$ that satisfies the equation (1) a.a. on $J \times \Omega$ and the initial conditions (2) are satisfied.

Let $h \in L^1(J, \mathbb{R}^n)$. We need the following lemma:

Lemma 2.9. [1, 4] A function $u \in AC(J, \mathbb{R}^n)$ is a solution of problem

$$\begin{cases} {}^c D_{\theta}^r u(x, y) = h(x, y); & \text{for a.a. } (x, y) \in J := [0, a] \times [0, b], \\ u(x, 0) = \varphi(x); & x \in [0, a], \\ u(0, y) = \psi(y); & y \in [0, b], \\ \varphi(0) = \psi(0). \end{cases}$$

if and only if u satisfies

$$u(x, y) = \mu(x, y) + I_{\theta}^r h(x, y); \text{ for a.a. } (x, y) \in J,$$

where

$$\mu(x, y) = \varphi(x) + \psi(y) - \varphi(0).$$

Let us assume that the function f is random Carathéodory on $J \times E \times \Omega$. From the above lemma, we have the following Lemma.

Lemma 2.10. Let $0 < r_1, r_2 \leq 1$. A function $u \in \Omega \times AC$ is a solution of the random fractional integral equation

$$u(x, y, w) = \mu(x, y, w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds, \quad (3)$$

where

$$\mu(x, y, w) = \varphi(x, w) + \psi(y, w) - \varphi(0, w),$$

if and only if u is a solution of the random problem (1)-(2).

Now, we consider the Ulam stability of fractional random differential equation (1). Let ϵ be a positive real number and $\Phi : J \times \Omega \rightarrow [0, \infty)$ be a measurable and bounded function. We consider the following inequalities

$$\|{}^c D_{\theta}^r u(x, y, w) - f(x, y, u(x, y, w), w)\|_E \leq \epsilon; \text{ for a.a. } (x, y) \in J, w \in \Omega. \quad (4)$$

$$\|{}^c D_{\theta}^r u(x, y, w) - f(x, y, u(x, y, w), w)\|_E \leq \Phi(x, y, w); \text{ for a.a. } (x, y) \in J, w \in \Omega. \quad (5)$$

$$\|{}^c D_{\theta}^r u(x, y, w) - f(x, y, u(x, y, w), w)\|_E \leq \epsilon \Phi(x, y, w); \text{ for a.a. } (x, y) \in J, w \in \Omega. \quad (6)$$

Definition 2.11. The random equation (1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \rightarrow AC(J)$ of the inequality (4), there exists a random solution $v : \Omega \rightarrow AC(J)$ of problem (1) with

$$\|u(x, y, w) - v(x, y, w)\|_E \leq \epsilon c_f; (x, y) \in J, w \in \Omega.$$

Definition 2.12. The random equation (1) is generalized Ulam-Hyers stable if there exists $\theta_f \in C([0, \infty), [0, \infty))$, $\theta_f(0) = 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \rightarrow AC(J)$ of the inequality (4), there exists a random solution $v : \Omega \rightarrow AC(J)$ of problem (1) with

$$\|u(x, y, w) - v(x, y, w)\|_E \leq \theta_f(\epsilon); (x, y) \in J, w \in \Omega.$$

Definition 2.13. The random equation (1) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each $\epsilon > 0$ and for each random solution $u : \Omega \rightarrow AC(J)$ of the inequality (6), there exists a random solution $v : \Omega \rightarrow AC(J)$ of problem (1) with

$$\|u(x, y, w) - v(x, y, w)\|_E \leq \epsilon c_{f,\Phi} \Phi(x, y, w); (x, y) \in J, w \in \Omega.$$

Definition 2.14. The random equation (1) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi} > 0$ such that for each random solution $u : \Omega \rightarrow AC(J)$ of the inequality (5), there exists a random solution $v : \Omega \rightarrow AC(J)$ of problem (1) with

$$\|u(x, y, w) - v(x, y, w)\|_E \leq c_{f,\Phi} \Phi(x, y, w); (x, y) \in J, w \in \Omega.$$

Remark 2.15. It is clear that

- (i) Definition 2.11 \Rightarrow Definition 2.12,
- (ii) Definition 2.13 \Rightarrow Definition 2.14,
- (iii) Definition 2.13 for $\Phi(x, y) = 1 \Rightarrow$ Definition 2.11.

Remark 2.16. A function $u : \Omega \rightarrow AC(J)$ is a solution of the inequality (4) if and only if there exists a function $g : \Omega \rightarrow C(J)$ (which depends on u) such that

- (i) $\|g(x, y, w)\|_E \leq \epsilon$,
- (ii) ${}^c D_\theta^r u(x, y, w) = f(x, y, u(x, y, w), w) + g(x, y, w)$; a.a. $(x, y) \in J, w \in \Omega$.

One can have similar remarks for the inequalities (5) and (6). So, the Ulam stabilities of the fractional random differential equations are some special types of data dependence of the solutions of fractional differential equations.

Lemma 2.17. [12] If Y is a bounded subset of Banach space X , then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^\infty \subset Y$ such that

$$\alpha(Y) \leq 2\alpha(\{y_k\}_{k=1}^\infty) + \epsilon.$$

Lemma 2.18. [31, 50] If $\{u_k\}_{k=1}^\infty \subset L^1(J)$ is uniformly integrable, then $\alpha(\{u_k\}_{k=1}^\infty)$ is measurable and for each $(x, y) \in J$,

$$\alpha\left(\left\{\int_0^x \int_0^y u_k(s, t) dt ds\right\}_{k=1}^\infty\right) \leq 2 \int_0^x \int_0^y \alpha(\{u_k(s, t)\}_{k=1}^\infty) dt ds.$$

Lemma 2.19. [30] Let F be a closed and convex subset of a real Banach space, let $G : F \rightarrow F$ be a continuous operator and $G(F)$ be bounded. If there exist a constant $k \in [0, 1)$ such that for each bounded subset $B \subset F$,

$$\alpha(G(B)) \leq k\alpha(B),$$

then G has a fixed point in F .

In the sequel we will make use of the following generalization of Gronwall's lemma.

Lemma 2.20. (Gronwall lemma) [32, 33] Let $v : J \times \Omega \rightarrow [0, \infty)$ be a real function and $\omega(x, y, w)$ be a measurable, nonnegative and locally integrable function on $J \times \Omega$. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that

$$v(x, y, w) \leq \omega(x, y, w) + c \int_0^x \int_0^y \frac{v(s, t, w)}{(x-s)^{r_1}(y-t)^{r_2}} dt ds,$$

then there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$v(x, y, w) \leq \omega(x, y, w) + \delta c \int_0^x \int_0^y \frac{\omega(s, t, w)}{(x-s)^{r_1}(y-t)^{r_2}} dt ds,$$

for every $(x, y) \in J$ and $w \in \Omega$.

3. Existence and stability Results

In this section, we discuss the existence of random solutions and we present conditions for the Ulam stability for the problem (1)-(2).

Lemma 3.1. If $u : \Omega \rightarrow AC(J)$ is a solution of the inequality (4) then u is a solution of the following integral inequality

$$\begin{aligned} & \left\| u(x, y, w) - \mu(x, y, w) - \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w) dt ds \right\|_E \\ & \leq \frac{\epsilon a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}; \text{ if } (x, y) \in J, w \in \Omega. \end{aligned} \quad (7)$$

Proof. By Remark 2.16, for $(x, y) \in J$ and $w \in \Omega$ there exists $g : \Omega \rightarrow C(J)$ such that

$${}^c D_\theta^r u(x, y, w) = f(x, y, u(x, y, w), w) + g(x, y, w).$$

Then, for each $(x, y) \in J$ and $w \in \Omega$, we get

$$u(x, y, w) = \mu(x, y, w) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} [g(s, t, w) + f(s, t, u(s, t, w), w)] dt ds.$$

Thus, for each $(x, y) \in J$ and $w \in \Omega$, we obtain

$$\begin{aligned} & \left\| u(x, y, w) - \mu(x, y, w) - \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w) dt ds \right\|_E \\ &= \left\| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} g(s, t, w) dt ds \right\|_E \\ &\leq \frac{\epsilon a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)}. \end{aligned}$$

Hence, we obtain (7). \square

Remark 3.2. One can obtain similar results for the solutions of the inequalities (5) and (6).

The following hypotheses will be used in the sequel.

- (H₁) The functions $w \mapsto \varphi(x, 0, w)$ and $w \mapsto \psi(0, y, w)$ are measurable and bounded for a.e. $x \in [0, a]$ and $y \in [0, b]$ respectively,
- (H₂) The function f is random Carathéodory on $J \times E \times \Omega$,
- (H₃) There exist functions $p_1, p_2 : J \times \Omega \rightarrow [0, \infty)$ with $p_i(\cdot, w) \in L^\infty(J, [0, \infty))$; $i = 1, 2$ such that for each $w \in \Omega$,

$$\|f(x, y, u, w)\|_E \leq p_1(x, y, w) + p_2(x, y, w)\|u\|_E,$$

for all $u \in E$ and a.e. $(x, y) \in J$,

- (H₄) There exists a function $q : J \times \Omega \rightarrow [0, \infty)$ with $q(\cdot, w) \in L^\infty(J, [0, \infty))$ for each $w \in \Omega$ such that for any bounded $B \subset E$,

$$\alpha(f(x, y, B, w)) \leq q(x, y, w)\alpha(B), \text{ for a.e. } (x, y) \in J,$$

- (H₅) There exists a random function $R : \Omega \rightarrow (0, \infty)$ such that

$$\mu^*(w) + \frac{(p_1^*(w) + p_2^*(w)R(w))a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \leq R(w),$$

where

$$\mu^*(w) = \sup_{(x,y) \in J} \|\mu(x, y, w)\|_E, \quad p_i^*(w) = \sup_{(x,y) \in J} \text{ess } p_i(x, y, w); \quad i = 1, 2,$$

(H_6) The function f satisfies

$$\|f(x, y, u, w) - f(x, y, \bar{u}, w)\|_E \leq q(x, y, w)\|u - \bar{u}\|_E,$$

for each $(x, y) \in J$, $w \in \Omega$ and $u, \bar{u} \in E$,

(H_7) $\Phi(w) \in L^1(J, [0, \infty))$ for all $w \in \Omega$, and there exists $\lambda_\Phi > 0$ such that, for each $(x, y) \in J$ we have

$$(I_\theta^r \Phi)(x, y, w) \leq \lambda_\Phi \Phi(x, y, w).$$

Remark 3.3.

1. Hypothesis (H_6) implies hypothesis (H_3), with

$$p_1(x, y, w) = \|f(x, y, 0, w)\|, \text{ and } p_2(x, y, w) = q(x, y, w).$$

2. Hypotheses (H_4) and (H_6) are equivalent ([9]).

Set

$$q^* = \sup_{(x, y, w) \in J \times \Omega} \text{ess } q(x, y, w).$$

Theorem 3.4. *Assume that hypotheses (H_1) – (H_5) hold. If*

$$\ell := \frac{4q^* a^{r_1} b^{r_2}}{\Gamma(1 + r_1)\Gamma(1 + r_2)} < 1,$$

then the problem (1)-(2) has a random solution defined on J .

Proof. From hypotheses (H_2), (H_3), for each $w \in \Omega$ and almost all $(x, y) \in J$, we have that $f(x, y, u(x, y, w), w)$ is in L^1 . By using Lemma 2.10, the problem (1)-(2) is equivalent to the integral equation

$$u(x, y, w) = \mu(x, y, w) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w) dt ds;$$

for each $w \in \Omega$ and a.e. $(x, y) \in J$.

Define the operator $N : \Omega \times \mathcal{C} \rightarrow \mathcal{C}$ by

$$(N(w)u)(x, y) = \mu(x, y, w) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w) dt ds.$$

Since the functions φ, ψ and f are absolutely continuous, then the function μ and the indefinite integral are absolutely continuous for all $w \in \Omega$ and almost all $(x, y) \in J$. Again, as the map μ is continuous for all $w \in \Omega$ and the indefinite integral is continuous on J , then $N(w)$ defines a mapping $N : \Omega \times \mathcal{C} \rightarrow \mathcal{C}$. Hence u is a solution for the problem (1)-(2) if and only if $u = (N(w))u$. We shall show that the operator N satisfies all conditions of Lemma 2.19. The proof will be given in several steps.

Step 1: $N(w)$ is a random operator with stochastic domain on \mathcal{C} .

Since $f(x, y, u, w)$ is random Carathéodory, the map $w \rightarrow f(x, y, u, w)$ is measurable in view of Definition 2.1. Similarly, the product $(x-s)^{r_1-1}(y-t)^{r_2-1}f(s, t, u(s, t, w), w)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$w \mapsto \mu(x, y, w) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s, t, u(s, t, w), w) dt ds,$$

is measurable. As a result, N is a random operator on $\Omega \times \mathcal{C}$ into \mathcal{C} .

Let $W : \Omega \rightarrow \mathcal{P}(\mathcal{C})$ be defined by

$$W(w) = \{u \in \mathcal{C} : \|u\|_\infty \leq R(w)\},$$

with $W(w)$ bounded, closed, convex and solid for all $w \in \Omega$. Then W is measurable by Lemma [[20], Lemma 17]. Let $w \in \Omega$ be fixed, then from (H_3) and (H_5) for any $u \in W(w)$, we get

$$\begin{aligned} & \|(N(w)u)(x, y)\|_E \\ & \leq \|\mu(x, y, w)\|_E + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(s, t, u(s, t, w), w)\|_E dt ds \\ & \leq \|\mu(x, y, w)\|_E + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_1(s, t, w) dt ds \\ & \quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} p_2(s, t, w) \|u(s, t, w)\|_E dt ds \\ & \leq \mu^*(w) + \frac{p_1^*(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dt ds \\ & \quad + \frac{p_2^*(w)R(w)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dt ds \\ & \leq \mu^*(w) + \frac{(p_1^*(w) + p_2^*(w)R(w))a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ & \leq R(w). \end{aligned}$$

Therefore, N is a random operator with stochastic domain W and $N(w) : W(w) \rightarrow W(w)$. Furthermore, $N(w)$ maps bounded sets into bounded sets in \mathcal{C} .

Step 2: $N(w)$ is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \rightarrow u$ in \mathcal{C} . Then, for each $(x, y) \in J$ and $w \in \Omega$, we have

$$\begin{aligned} & \|(N(w)u_n)(x, y) - (N(w)u)(x, y)\|_E \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ & \quad \times \|f(s, t, u_n(s, t, w), w) - f(s, t, u(s, t, w), w)\|_E dt ds. \end{aligned}$$

Using the Lebesgue Dominated Convergence Theorem, we get

$$\|N(w)u_n - N(w)u\|_\infty \rightarrow 0 \text{ as } n \rightarrow \infty.$$

As a consequence of Steps 1 and 2, we can conclude that $N(w) : W(w) \rightarrow W(w)$ is a continuous random operator with stochastic domain W , and $N(w)(W(w))$ is bounded.

Step 3: For each bounded subset B of $W(w)$ we have

$$\alpha(N(w)B) \leq \ell\alpha(B).$$

Let $w \in \Omega$ be fixed. From Lemmas 2.17 and 2.18, for any $B \subset W$ and any $\epsilon > 0$, there exists a sequence $\{u_n\}_{n=1}^\infty \subset B$, such that for all $(x, y) \in J$, we have

$$\begin{aligned} & \alpha((N(w)B)(x, y)) \\ &= \alpha\left(\left\{\mu(x, y, w) + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w) dt ds; u \in B\right\}\right) \\ &\leq 2\alpha\left(\left\{\int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u_n(s, t, w), w) dt ds\right\}_{n=1}^\infty\right) + \epsilon \\ &\leq 4 \int_0^x \int_0^y \alpha\left(\left\{\frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u_n(s, t, w), w)\right\}_{n=1}^\infty\right) dt ds + \epsilon \\ &\leq 4 \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \alpha(\{f(s, t, u_n(s, t, w), w)\}_{n=1}^\infty) dt ds + \epsilon \\ &\leq 4 \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} q(s, t, w) \alpha(\{u_n(s, t, w)\}_{n=1}^\infty) dt ds + \epsilon \\ &\leq \left(4 \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} q(s, t, w) ds dt\right) \alpha(\{u_n\}_{n=1}^\infty) + \epsilon \\ &\leq \left(4 \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} q(s, t, w) dt ds\right) \alpha(B) + \epsilon \\ &\leq \frac{4q^* a^{r_1} b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \alpha(B) + \epsilon \\ &= \ell\alpha(B) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then

$$\alpha(N(B)) \leq \ell\alpha(B).$$

It follows from Lemma 2.19 that for each $w \in \Omega$, N has at least one fixed point in W . Since $\bigcap_{w \in \Omega} \text{int}W(w) \neq \emptyset$ the hypothesis that a measurable selector of $\text{int}W$ exists holds. By Lemma 2.19, N has a stochastic fixed point, i.e., the problem (1)-(2) has at least one random solution on \mathcal{C} . \square

Theorem 3.5. *Assume that the assumptions $(H_1), (H_2), (H_5) - (H_7)$ hold. Then the random equation (1) is generalized Ulam-Hyers-Rassias stable.*

Proof. Let $u : \Omega \rightarrow AC(J)$ be a solution of the inequality (5). By Theorem 3.4, there exists v which is a solution of the random problem (1)-(2). Hence

$$v(x, y, w) = \mu(x, y, w)$$

$$+ \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, v(s, t, w), w) dt ds; \quad (x, y) \in J, \quad w \in \Omega.$$

By differential inequality (5), for each $(x, y) \in J$ and $w \in \Omega$, we have

$$\begin{aligned} & \left\| u(x, y, w) - \mu(x, y, w) - \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w) dt ds \right\|_E \\ & \leq \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \Phi(s, t, w) dt ds. \end{aligned}$$

Thus, by (H_7) for each $(x, y) \in J$ and $w \in \Omega$, we obtain

$$\begin{aligned} & \left\| u(x, y, w) - \mu(x, y, w) - \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w) dt ds \right\|_E \\ & \leq \lambda_\Phi \Phi(x, y, w). \end{aligned}$$

Hence for each $(x, y) \in J$ and $w \in \Omega$, it follows that

$$\begin{aligned} & \|u(x, y, w) - v(x, y, w)\|_E \\ & \leq \left\| u(x, y, w) - \mu(x, y, w) - \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(s, t, u(s, t, w), w) dt ds \right\|_E \\ & \quad + \int_0^x \int_0^y \frac{(x-s)^{r_1-1}(y-t)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \|f(s, t, u(s, t, w), w) - f(s, t, v(s, t, w), w)\|_E dt ds. \end{aligned}$$

From (H_6) , for each $(x, y) \in J$ and $w \in \Omega$, we get

$$\begin{aligned} & \|u(x, y, w) - v(x, y, w)\|_E \\ & \leq \lambda_\Phi \Phi(x, y, w) + \frac{q^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|u(s, t, w) - v(s, t, w)\|_E dt ds. \end{aligned}$$

From Lemma 2.20, there exists a constant $\delta = \delta(r_1, r_2)$ such that

$$\begin{aligned} \|u(x, y, w) - v(x, y, w)\|_E & \leq \lambda_\Phi \Phi(x, y, w) \\ & \quad + \frac{\delta q^* \lambda_\Phi}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \Phi(s, t, w) dt ds \\ & \leq (1 + \delta q^* \lambda_\Phi) \lambda_\Phi \Phi(x, y, w) \\ & := c_{f, \Phi} \Phi(x, y, w). \end{aligned}$$

Finally, the random equation (1) is generalized Ulam-Hyers-Rassias stable. \square

4. An Example

Let $E = \mathbb{R}$, $\Omega = (-\infty, 0)$ be equipped with the usual σ -algebra consisting of Lebesgue measurable subsets of $(-\infty, 0)$. Given a measurable function $u : \Omega \rightarrow AC([0, 1] \times [0, 1])$, consider the following partial functional random differential equation of the form

$$({}^c D_\theta^\alpha u)(x, y, w) = \frac{w^2 e^{-x-y-10}}{1+w^2+|u(x, y, w)|}; \quad \text{a.a. } (x, y) \in J = [0, 1] \times [0, 1], \quad w \in \Omega, \quad (8)$$

with the initial conditions

$$\begin{cases} u(x, 0, w) = x \sin w; & x \in [0, 1], \\ u(0, y, w) = y^2 \cos w; & y \in [0, 1], \end{cases} \quad w \in \Omega, \quad (9)$$

where $(r_1, r_2) \in (0, 1] \times (0, 1]$. Set

$$f(x, y, u(x, y, w), w) = \frac{w^2}{(1 + w^2 + |u(x, y, w)|)e^{x+y+10}}, \quad (x, y) \in [0, 1] \times [0, 1], \quad w \in \Omega.$$

The functions $w \mapsto \varphi(x, 0, w) = x \sin w$ and $w \mapsto \psi(0, y, w) = y^2 \cos w$ are measurable and bounded with

$$|\varphi(x, 0, w)| \leq 1, \quad |\psi(0, y, w)| \leq 1,$$

hence, the condition (H_1) is satisfied.

Clearly, the map $(x, y, w) \mapsto f(x, y, u, w)$ is jointly continuous for all $u \in \mathbb{R}$ and hence jointly measurable for all $u \in \mathbb{R}$. Also the map $u \mapsto f(x, y, u, w)$ is continuous for all $(x, y) \in J$ and $w \in \Omega$. So the function f is Carathéodory on $[0, 1] \times [0, 1] \times \mathbb{R} \times \Omega$.

For each $u \in \mathbb{R}$, $(x, y) \in [0, 1] \times [0, 1]$ and $w \in \Omega$, we have

$$|f(x, y, u, w)| \leq 1 + \frac{1}{e^{10}}|u|.$$

Hence the condition (H_3) is satisfied with $p_1(x, y, w) = p_1^* = 1$ and $p_2(x, y, w) = p_2^* = \frac{1}{e^{10}}$.

Also, condition (H_6) is satisfied with $q^* = \frac{1}{e^{10}}$. Indeed, for each $u, \bar{u} \in \mathbb{R}$ and $(x, y) \in [0, 1] \times [0, 1]$ and $w \in \Omega$, we get

$$|f(x, y, u, w) - f(x, y, \bar{u}, w)| \leq \frac{1}{e^{10}}|u - \bar{u}|.$$

We shall show that condition $\ell < 1$ holds with $a = b = 1$. Indeed, we have $q^* = \frac{1}{e^{10}}$ and for each $(r_1, r_2) \in (0, 1] \times (0, 1]$ we get

$$\begin{aligned} \ell &= \frac{4q^*a^{r_1}b^{r_2}}{\Gamma(1+r_1)\Gamma(1+r_2)} \\ &= \frac{4}{e^{10}\Gamma(1+r_1)\Gamma(1+r_2)} \\ &< 1. \end{aligned}$$

Finally, the hypothesis (H_7) is satisfied with

$$\Phi(x, y, w) = w^2xy^2,$$

and

$$\lambda_\Phi = \frac{2}{\Gamma(2+r_1)\Gamma(3+r_2)}.$$

Indeed, for each $(x, y) \in [0, 1] \times [0, 1]$ and $w \in \Omega$, we get

$$\begin{aligned} (I_{\theta}^r \Phi)(x, y, w) &= \frac{w^2 \Gamma(2) \Gamma(3)}{\Gamma(2 + r_1) \Gamma(3 + r_2)} x^{1+r_1} y^{2+r_2} \\ &\leq \frac{2w^2 xy^2}{\Gamma(2 + r_1) \Gamma(3 + r_2)} \\ &= \lambda_{\Phi} \Phi(x, y, w). \end{aligned}$$

Consequently, Theorem 3.4 implies that the problem (8)-(9) has a random solution defined on $[0, 1] \times [0, 1]$, and Theorem 3.5 implies that the random equation (8) is generalized Ulam-Hyers-Rassias stable.

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Measure of Noncompactness and Neutral Functional Differential Equations with State-Dependent Delay

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ABSTRACT: Our aim in this work is to study the existence of solutions of first and second order for neutral functional differential equations with state-dependent delay. We use the Mönch's fixed point theorem for the existence of solutions and the concept of measures of noncompactness.

AMS Subject Classification: 34G20, 34K20, 34K30.

Keywords and Phrases: Neutral functional differential equation; Mild solution; Infinite delay; State-dependent delay; Fixed point; Semigroup theory; Cosine function; Measure of noncompactness.

1. Introduction

In this work we prove the existence of solutions of first and second order for neutral functional differential equation with state-dependent delay. Our investigations will be situated on the Banach space of real valued functions which are defined, continuous and bounded on a real axis \mathbb{R} . More precisely, we will consider the following problem

$$\frac{d}{dt}[y(t) - g(t, y_{\rho(t, y_t)})] = A[y(t) - g(t, y_{\rho(t, y_t)})] + f(t, y_{\rho(t, y_t)}), \quad \text{a.e. } t \in J := [0, +\infty) \quad (1)$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \quad (2)$$

where $f, g : J \times \mathcal{B} \rightarrow E$ are given functions, $A : D(A) \subset E \rightarrow E$ is the infinitesimal generator of a strongly continuous semigroup $T(t), t \in J$, \mathcal{B} is the phase space to be specified later, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \rightarrow (-\infty, +\infty)$ and $(E, |\cdot|)$ is a real Banach space. For any function y defined on $(-\infty, +\infty)$ and any $t \in J$ we denote by y_t the element of \mathcal{B} defined by $y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the state up to the present time t . We assume that the histories y_t to some abstract phases \mathcal{B} , to be specified later.

Later, we consider the following problem

$$\frac{d}{dt}[y'(t) - g(t, y_{\rho(t, y_t)})] = Ay(t) + f(t, y_{\rho(t, y_t)}), \quad \text{a.e. } t \in J := [0, +\infty) \quad (3)$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \quad y'(0) = \varphi, \quad (4)$$

where $f, g : J \times \mathcal{B} \rightarrow E$ is given function, $A : D(A) \subset E \rightarrow E$ is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t \in \mathbb{R}}$, on E , $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \rightarrow (-\infty, +\infty)$, and $(E, |\cdot|)$ is a real Banach space. For the both problems, we will use Mönch's fixed theorem and the concept of measures of noncompactness combined with the Corduneanu's compactness criteria.

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years. The literature devoted to this subject is concerned fundamentally with first order functional differential equations for which the state belong to some finite dimensional space, see among another works [2, 11, 13, 15, 19, 20, 21].

The problem of the existence of solutions for first and second order partial functional differential with state-dependent delay have treated recently in [5, 10, 24, 32, 33, 37]. The literature relative second order differential system with state-dependent delay is very restrict, and related this matter we only cite [34] for ordinary differential system and [22] for abstract partial differential systems. Recently, in [1, 6, 8, 12] the authors provided some global existence and stability results for various classes of functional evolution equations with delay in Banach and Fréchet spaces.

The cosine function theory is related to abstract linear second order differential equations in the same manner that the semigroup theory of bounded linear operators is related to first order partial differential equations and its equally appealing devoted their generality and simplicity. For basic concepts and applications of this theory, we refer to the reader to Fattorini [16], Travis and Webb [36].

Our purpose in this work is consider a simultaneous generalization of the classical second order abstract Cauchy problem studied by Travis and Webb in [35, 36]. Additionally, we observe that the ideas and techniques in this paper permit the reformulation of the problems studied in [7, 28] to the context of partial second order differential equations, see [35] pp. 557 and the referred papers for details.

In this paper we use the technique of measures of noncompactness. It is well known that this method provides an excellent tool for obtaining existence of solutions of nonlinear differential equation. This technique works fruitfully for both integral and differential equations. More details are found in Akhmerov *et al.* [3], Alvéares [4], Banaś and Goebel [9], Guo *et al.* [17], Mönch [29], Mönch and Von Harten [30], and the references therein.

The literature on neutral functional evolution equations with delay on unbounded intervals is very limited. Some of them are stated in the Fréchet space setting, while the present ones are stated in the Banach setting. In particular our results extend those considered on bounded intervals by Hernandez and Mckibben [23]. Thus, the present paper complements that study.

2. Preliminaries

In this section we present briefly some notations and definition, and theorem which are used throughout this work.

In this paper, we will employ an axiomatic definition of the phase space \mathcal{B} introduced by Hale and Kato in [18] and follow the terminology used in [26]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E , and satisfying the following axioms :

- (A₁) If $y : (-\infty, b) \rightarrow E, b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold :
- (i) $y_t \in \mathcal{B}$;
 - (ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
 - (iii) There exist two functions $L(\cdot), M(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ independent of y with L continuous and bounded, and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq L(t) \sup\{|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A₂) For the function y in (A₁), y_t is a \mathcal{B} -valued continuous function on J .

(A₃) The space \mathcal{B} is complete.

Denote

$$l = \sup\{L(t) : t \in J\},$$

and

$$m = \sup\{M(t) : t \in J\}.$$

Remark 2.1.

1. (ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.
2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.
3. From the equivalence of in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$: We necessarily have that $\phi(0) = \psi(0)$.

Example 2.2. (The phase space $(\mathbf{C}_r \times \mathbf{L}^p(\mathbf{g}, \mathbf{E}))$)

Let $g : (-\infty, -r) \rightarrow \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and locally bounded function γ on $(-\infty, 0]$ such that $g(\xi + \theta) \leq \gamma(\xi)g(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_{\xi}$, where $N_{\xi} \subseteq (-\infty, -r)$ is a set with zero Lebesgue's measure. The space $C_r \times L^p(g, E)$ consists of all classes of functions $\varphi : (-\infty, 0] \rightarrow \mathbb{R}$ such that ϕ is continuous on $[-r, 0]$, Lebesgue-measurable and $g\|\phi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(g, E)$ is defined by

$$\|\phi\|_{\mathcal{B}} := \sup\{\|\phi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} g(\theta)\|\phi(\theta)\|^p d\theta \right)^{\frac{1}{p}}.$$

Assume that $g(\cdot)$ verifies the condition $(g-5)$, $(g-6)$ and $(g-7)$ in the nomenclature [26]. In this case, $\mathcal{B} = C_r \times L^p(g, E)$ verifies assumptions (A_1) , (A_2) , (A_3) see ([26] Theorem 1.3.8) for details. Moreover, when $r = 0$ and $p = 2$ we have that

$$H = 1, \quad M(t) = \gamma(-t)^{\frac{1}{2}}, \quad L(t) = 1 + \left(\int_{-t}^0 g(\theta) d\theta \right)^{\frac{1}{2}}, \quad t \geq 0.$$

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E .

By $BC := BC(-\infty, +\infty)$ we denote the Banach space of all bounded and continuous functions from $(-\infty, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC} = \sup_{t \in (-\infty, +\infty)} |y(t)|.$$

Finally, by $BC' := BC([0, +\infty))$ we denote the Banach space of all bounded and continuous functions from $[0, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC'} = \sup_{t \in [0, +\infty)} |y(t)|.$$

Definition 2.3. A map $f : J \times \mathcal{B} \rightarrow E$ is said to be Carathéodory if

- (i) $t \rightarrow f(t, y)$ is measurable for all $y \in \mathcal{B}$.
- (ii) $y \rightarrow f(t, y)$ is continuous for almost each $t \in J$.

Now let us recall some fundamental facts of the Kuratowski measure of noncompactness.

Definition 2.4. Let E be a Banach space and Ω_E be the family of bounded subsets of E . The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \rightarrow [0, \infty)$ defined by

$$\alpha(B) = \inf \{ \epsilon > 0 : B \subseteq \bigcup_{i=1}^n B_i \text{ and } \text{diam}(B_i) \leq \epsilon \}; \text{ here } B \in \Omega_E.$$

The Kuratowski measure of noncompactness satisfies the following properties

- (a) $\alpha(B) = 0 \Leftrightarrow \bar{B}$ is compact (B is relatively compact).
- (b) $\alpha(B) = \alpha(\bar{B})$.
- (c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.
- (d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.
- (e) $\alpha(cB) = |c| \alpha(B)$; $c \in \mathbb{R}$.
- (f) $\alpha(\text{conv} B) = \alpha(B)$.

Theorem 2.5. (Mönch fixed point)[29]

Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication

$$V = \overline{\text{conv}N(V)} \text{ or } V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D , then N has a fixed point.

Lemma 2.6. (Corduneanu)[14]

Let $D \subset BC([0, +\infty), E)$. Then D is relatively compact if the following conditions hold:

- (a) D is bounded in BC .
- (b) The function belonging to D is almost equicontinuous on $[0, +\infty)$, i.e., equicontinuous on every compact of $[0, +\infty)$.
- (c) The set $D(t) := \{y(t) : y \in D\}$ is relatively compact on every compact of $[0, +\infty)$.
- (d) The function from D is equiconvergent, that is, given $\epsilon > 0$, responds $T(\epsilon) > 0$ such that $|u(t) - \lim_{t \rightarrow +\infty} u(t)| < \epsilon$, for any $t \geq T(\epsilon)$ and $u \in D$.

3. The first order problem

In this section we give our main existence result for problem (1)-(2). Before starting and proving this result, we give the definition of the mild solution.

Definition 3.1. We say that a continuous function $y : (-\infty, +\infty) \rightarrow E$ is a mild solution of problem (1)-(2) if $y(t) = \phi(t)$, $t \in (-\infty, 0]$ and the restriction of $y(\cdot)$ to the interval $[0, +\infty)$ is continuous and satisfies the following integral equation:

$$y(t) = T(t)[\phi(0) - g(0, \phi(0))] + g(t, y_{\rho(t, y_t)}) + \int_0^t T(t-s)f(s, y_{\rho(s, y_s)})ds, \quad t \in J. \quad (5)$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s, \phi) : (s, \phi) \in J \times \mathcal{B}, \rho(s, \phi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \rightarrow \mathbb{R}$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) The function $t \rightarrow \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into \mathcal{B} and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \rightarrow (0, \infty)$ such that

$$\|\phi_t\| \leq \mathcal{L}^\phi(t)\|\phi\| \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

Remark 3.2. The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see for instance [26].

Lemma 3.3. ([25]) *If $y : (-\infty, +\infty) \rightarrow E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + l \sup\{|y(\theta)|; \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where $\mathcal{L}^\phi = \sup_{t \in \mathcal{R}(\rho^-)} \mathcal{L}^\phi(t)$.

Let us introduce the following hypotheses

(H₁) $A : D(A) \subset E \rightarrow E$ is the infinitesimal generator of a uniformly continuous semigroup $T(t)$, $t \in J$. Let $M' = \sup\{\|T\|_{\mathcal{B}(E)} : t \geq 0\}$.

(H₂) The function $f : J \times \mathcal{B} \rightarrow E$ is Carathéodory.

(H₃) There exists a continuous function $k : J \rightarrow [0, +\infty)$ such that:

$$|f(t, u)| \leq k(t)\|u\|_{\mathcal{B}}, \quad t \in J, \quad u \in \mathcal{B}$$

and

$$k^* := \sup_{t \in J} \int_0^t k(s) ds < \infty.$$

(H₄) For each bounded set $B \subset \mathcal{B}$, and each $t \in J$ we have

$$\alpha(f(t, B)) \leq k(t)\alpha(B).$$

(H₅) The function $g : J \times \mathcal{B} \rightarrow E$ is Carathéodory there exists a continuous function $k_g : J \rightarrow [0, +\infty)$ such that

$$|g(t, u)| \leq k_g(t)\|u\|_{\mathcal{B}}, \quad \text{for each } u \in \mathcal{B}$$

and

$$k_g^* := \sup_{t \in J} \int_0^t k_g(s) ds < \infty.$$

(H₆) For each bounded set $B \subset \mathcal{B}$, and each $t \in J$ we have

$$\alpha(g(t, B)) \leq k_g(t)\alpha(B).$$

(H₇) For any bounded set $B \subset \mathcal{B}$, the function $\{t \rightarrow g(t, y_t) : y \in B\}$ is equicontinuous on each compact interval of $[0, +\infty)$.

Theorem 3.4. *Assume that (H₁) – (H₇) and (H_φ) hold. If*

$$l(M'k^* + k_g) < 1, \tag{6}$$

then the problem (1)-(2) has at least one mild solution on BC .

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the operator $N : BC \rightarrow BC$ defined by:

$$(Ny)(t) = \begin{cases} \phi(t); & \text{if } t \in (-\infty, 0], \\ T(t) [\phi(0) - g(0, \phi(0))] \\ + g(t, y_{\rho(t, y_t)}) + \int_0^t T(t-s) f(s, y_{\rho(s, y_s)}) ds; & \text{if } t \in J. \end{cases}$$

Let $x(\cdot) : (-\infty, +\infty) \rightarrow E$ be the function defined by:

$$x(t) = \begin{cases} \phi(t); & \text{if } t \in (-\infty, 0]; \\ T(t) \phi(0); & \text{if } t \in J, \end{cases}$$

then $x_0 = \phi$. For each $z \in BC$ with $z(0) = 0$, we denote by \bar{z} the function

$$\bar{z}(t) = \begin{cases} 0; & \text{if } t \in (-\infty, 0]; \\ z(t); & \text{if } t \in J. \end{cases}$$

If y satisfies (5), we can decompose it as $y(t) = z(t) + x(t)$, $t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$ and the function $z(\cdot)$ satisfies

$$\begin{aligned} z(t) &= g(t, z_{\rho(t, z_t+x_t)} + x_{\rho(t, z_t+x_t)}) - T(t)g(0, \phi(0)) \\ &+ \int_0^t T(t-s)f(s, z_{\rho(s, z_s+x_s)} + x_{\rho(s, z_s+x_s)})ds, t \in J. \end{aligned}$$

Set

$$BC'_0 = \{z \in BC' : z(0) = 0\}$$

and let

$$\|z\|_{BC'_0} = \sup\{|z(t)| : t \in J\}, z \in BC'_0.$$

BC'_0 is a Banach space with the norm $\|\cdot\|_{BC'_0}$. We define the operator $\mathcal{A} : BC'_0 \rightarrow BC'_0$ by:

$$\begin{aligned} \mathcal{A}(z)(t) &= g(t, z_{\rho(t, z_t+x_t)} + x_{\rho(t, z_t+x_t)}) - T(t)g(0, \phi(0)) \\ &+ \int_0^t T(t-s)f(s, z_{\rho(s, z_s+x_s)} + x_{\rho(s, z_s+x_s)})ds, t \in J. \end{aligned}$$

We shall show that the operator \mathcal{A} satisfies all conditions of Mönch's fixed point theorem. The operator \mathcal{A} maps BC'_0 into BC'_0 , indeed the map $\mathcal{A}(z)$ is continuous

on $[0, +\infty)$ for any $z \in BC'_0$, and for each $t \in J$ we have

$$\begin{aligned}
|\mathcal{A}(z)(t)| &\leq |g(t, z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)})| + M'|g(0, \phi(0))| \\
&+ M' \int_0^t |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds \\
&\leq M'(k_g \|\phi\|_{\mathcal{B}}) + k_g \|z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)}\|_{\mathcal{B}} \\
&+ M' \int_0^t k(s) \|z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\|_{\mathcal{B}} ds \\
&\leq M'(k_g \|\phi\|_{\mathcal{B}}) + k_g(l|z(t)| + (m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}}) \\
&+ M' \int_0^t k(s)(l|z(s)| + (m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}}) ds.
\end{aligned}$$

Set

$$C_1 := (m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}}.$$

$$C_2 := M'(k_g \|\phi\|_{\mathcal{B}}) + k_g(m + \mathcal{L}^\phi + lM'H)\|\phi\|_{\mathcal{B}}.$$

Then, we have

$$\begin{aligned}
|\mathcal{A}(z)(t)| &\leq C_2 + k_g l|z(t)| + M'C_1 \int_0^t k(s) ds + M' \int_0^t l|z(s)|k(s) ds \\
&\leq C_2 + k_g l\|z\|_{BC'_0} + M'C_1 k^* + M'l\|z\|_{BC'_0} k^*.
\end{aligned}$$

Hence, $\mathcal{A}(z) \in BC'_0$.

Moreover, let $r > 0$ be such that

$$r \geq \frac{C_2 + M'C_1 k^*}{1 - l(M'k^* + k_g)},$$

and B_r be the closed ball in BC'_0 centered at the origin and of radius r . Let $z \in B_r$ and $t \in [0, +\infty)$. Then

$$|\mathcal{A}(z)(t)| \leq C_2 + k_g l r + M'C_1 k^* + M'k^* l r.$$

Thus

$$\|\mathcal{A}(z)\|_{BC'_0} \leq r,$$

which means that the operator \mathcal{A} transforms the ball B_r into itself.

Now we prove that $\mathcal{A} : B_r \rightarrow B_r$ satisfies the assumptions of Mönch's fixed theorem. The proof will be given in several steps.

Step 1: \mathcal{A} is continuous in B_r .

Let $\{z_n\}$ be a sequence such that $z_n \rightarrow z$ in B_r . At the first, we study the convergence

of the sequences $(z_{\rho(s, z_s^n)}^n)_{n \in \mathbb{N}}, s \in J$.

If $s \in J$ is such that $\rho(s, z_s) > 0$, then we have,

$$\begin{aligned} \|z_{\rho(s, z_s^n)}^n - z_{\rho(s, z_s)}\|_{\mathcal{B}} &\leq \|z_{\rho(s, z_s^n)}^n - z_{\rho(s, z_s^n)}\|_{\mathcal{B}} + \|z_{\rho(s, z_s^n)} - z_{\rho(s, z_s)}\|_{\mathcal{B}} \\ &\leq l \|z_n - z\|_{B_r} + \|z_{\rho(s, z_s^n)} - z_{\rho(s, z_s)}\|_{\mathcal{B}}, \end{aligned}$$

which proves that $z_{\rho(s, z_s^n)}^n \rightarrow z_{\rho(s, z_s)}$ in \mathcal{B} as $n \rightarrow \infty$ for every $s \in J$ such that $\rho(s, z_s) > 0$. Similarly, if $\rho(s, z_s) < 0$, we get

$$\|z_{\rho(s, z_s^n)}^n - z_{\rho(s, z_s)}\|_{\mathcal{B}} = \|\phi_{\rho(s, z_s^n)}^n - \phi_{\rho(s, z_s)}\|_{\mathcal{B}} = 0$$

which also shows that $z_{\rho(s, z_s^n)}^n \rightarrow z_{\rho(s, z_s)}$ in \mathcal{B} as $n \rightarrow \infty$ for every $s \in J$ such that $\rho(s, z_s) < 0$. Combining the previous arguments, we can prove that $z_{\rho(s, z_s)}^n \rightarrow \phi$ for every $s \in J$ such that $\rho(s, z_s) = 0$. Finally,

$$\begin{aligned} &|\mathcal{A}(z_n)(t) - \mathcal{A}(z)(t)| \\ &\leq |g(t, z_{\rho(t, z_t^n)}^n + x_{\rho(t, z_t^n)} + x_t) - g(t, z_{\rho(t, z_t)} + x_{\rho(t, z_t)} + x_t)| \\ &+ M' \int_0^t |f(s, z_{\rho(s, z_s^n)}^n + x_{\rho(s, z_s^n)} + x_s) - f(s, z_{\rho(s, z_s)} + x_{\rho(s, z_s)} + x_s)| ds \\ &\leq |g(t, z_{\rho(t, z_t^n)}^n + x_{\rho(t, z_t^n)} + x_t) - g(t, z_{\rho(t, z_t)} + x_{\rho(t, z_t)} + x_t)| \\ &+ M' \int_0^t |f(s, z_{\rho(s, z_s^n)}^n + x_{\rho(s, z_s^n)} + x_s) - f(s, z_{\rho(s, z_s)} + x_{\rho(s, z_s)} + x_s)| ds. \end{aligned}$$

Then by $(H_2), (H_5)$ we have

$$f(s, z_{\rho(s, z_s^n)}^n + x_{\rho(s, z_s^n)} + x_s) \rightarrow f(s, z_{\rho(s, z_s)} + x_{\rho(s, z_s)} + x_s), \text{ as } n \rightarrow \infty,$$

$$g(t, z_{\rho(t, z_t^n)}^n + x_{\rho(t, z_t^n)} + x_t) \rightarrow g(t, z_{\rho(t, z_t)} + x_{\rho(t, z_t)} + x_t), \text{ as } n \rightarrow \infty,$$

and by the Lebesgue dominated convergence theorem we get,

$$\|\mathcal{A}(z_n) - \mathcal{A}(z)\|_{BC'_0} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus \mathcal{A} is continuous.

Step 2 : $\mathcal{A}(B_r) \subset B_r$. This is clear.

Step 3 : $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0, b]$ of $[0, +\infty)$ for

$b > 0$. Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, we have:

$$\begin{aligned}
& |\mathcal{A}(z)(\tau_2) - \mathcal{A}(z)(\tau_1)| \\
& \leq |g(\tau_2, z_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})} + x_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})}) - g(\tau_1, z_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})} + x_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})})| \\
& + \|T(\tau_2) - T(\tau_1)\|_{B(E)} |g(0, \phi(0))| \\
& + \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds \\
& + \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds \\
& \leq |g(\tau_2, z_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})} + x_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})}) - g(\tau_1, z_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})} + x_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})})| \\
& + \|T(\tau_2) - T(\tau_1)\|_{B(E)} (k_g \|\phi\|_{\mathcal{B}}) \\
& + \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds \\
& + \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds \\
& \leq |g(\tau_2, z_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})} + x_{\rho(\tau_2, z_{\tau_2} + x_{\tau_2})}) - g(\tau_1, z_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})} + x_{\rho(\tau_1, z_{\tau_1} + x_{\tau_1})})| \\
& + \|T(\tau_2) - T(\tau_1)\|_{B(E)} (k_g \|\phi\|_{\mathcal{B}}) \\
& + C_1 \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} k(s) ds \\
& + rl \int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} k(s) ds \\
& + C_1 \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} k(s) ds \\
& + rl \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} k(s) ds.
\end{aligned}$$

When $\tau_1 \rightarrow \tau_2$, the right-hand side of the above inequality tends to zero, since $T(t)$ is uniformly continuous operator (see [31]) and since (H_7) , this proves the equicontinuity.

Step 4: $\mathcal{A}(B_r)$ is equiconvergent.

Let $t \in [0, +\infty)$ and $z \in B_r$, we have,

$$\begin{aligned}
|\mathcal{A}(z)(t)| & \leq |g(t, z_{\rho(t, z_t + x_t)} + x_{\rho(t, z_t + x_t)})| + M' |g(0, \phi(0))| \\
& + M' \int_0^t |f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)})| ds \\
& \leq C_2 + k_g lr + M' C_1 \int_0^t k(s) ds + M' rl \int_0^t k(s) ds.
\end{aligned}$$

Set

$$C_3 = C_2 + k_g lr + M' C_1 k^* + M' lr k^*.$$

Then we have

$$\lim_{t \rightarrow +\infty} |\mathcal{A}(z)(t)| \leq C_3.$$

Hence,

$$|\mathcal{A}(z)(t) - \mathcal{A}(z)(+\infty)| \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

Now let V be a subset of B_r such that $V \subset \overline{\text{conv}}(\mathcal{A}(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \rightarrow v(t) = \alpha(V(t))$ is continuous on \mathbb{R} .

$$\begin{aligned} V(t) &\leq \alpha(\mathcal{A}(V)(t) \cup \{0\})\alpha(\mathcal{A}(V)(t)) \\ &\leq k_g \alpha(V(t)) + M' \int_0^t k(s) \alpha(V(s)) ds \\ &\leq k_g v(t) + M' \int_0^t k(s) v(s) ds \\ &\leq l(k_g + M' k^*) \|v\|_\infty. \end{aligned}$$

This means that

$$\|v\|_\infty (1 - l(k^* M' + k_g)) \leq 0.$$

By (6) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$ and then $V(t)$ is relatively compact in E . As a consequence of Steps 1-4, with Lemma 2.6, and from Mönch's theorem, we deduce that \mathcal{A} has a fixed point z^* . Then $y^* = z^* + x$ is a fixed point of the operators N , which is a mild solution of the problem (1)-(2). \square

4. The Second order problem

In this section we are going to study existence of mild solution for problem (3)-(4). Before we mention a few results and notations respect of the cosine function theory which are needed to establish our results. Along of this section, A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t \geq 0}$ on Banach space $(E, |\cdot|)$. We denote by $(S(t))_{t \geq 0}$ the sine function associated with $(C(t))_{t \geq 0}$ which is defined by $S(t)y = \int_0^t C(s)y ds$, for $y \in E$ and $t \geq 0$.

The notation $[D(A)]$ stands for the domain of the operator A endowed with the graph norm $\|y\|_A = |y| + |Ay|$, $y \in D(A)$. Moreover, in this work, X is the space formed by the vector $y \in E$ for which $C(\cdot)y$ is of class C^1 on \mathbb{R} . It was proved by Kisyański [27] that X endowed with the norm

$$\|y\|_X = |y| + \sup_{0 \leq t \leq 1} |AS(t)y|, \quad y \in X,$$

is a Banach space. The operator valued function

$$G(t) = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix}$$

is a strongly continuous group of bounded linear operators on the space $X \times E$ generated by the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

defined on $D(A) \times X$. It follows this that $AS(t) : X \rightarrow E$ is a bounded linear operator and that $AS(t)y \rightarrow 0, t \rightarrow 0$, for each $y \in X$. Furthermore, if $y : [0, +\infty) \rightarrow E$ is a locally integrable function, then $z(t) = \int_0^t S(t-s)y(s)ds$ defined an X -valued continuous function. This is a consequence of the fact that:

$$\int_0^t G(t-s) \begin{pmatrix} 0 \\ y(s) \end{pmatrix} ds = \begin{pmatrix} \int_0^t S(t-s)y(s)ds \\ \int_0^t C(t-s)y(s)ds \end{pmatrix}$$

defines an $X \times E$ - valued continuous function. The existence of solutions for the second order abstract Cauchy problem.

$$\begin{cases} y''(t) = Ay(t) + h(t), & t \in J := [0, +\infty); \\ y(0) = y_0, & y'(0) = y_1, \end{cases} \quad (7)$$

where $h : J \rightarrow E$ is an integrable function has been discussed in [35]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [36].

Definition 4.1. The function $y(\cdot)$ given by:

$$y(t) = C(t)y_0 + S(t)y_1 + \int_0^t S(t-s)h(s)ds, t \in J,$$

is called mild solution of (7).

Remark 4.2. When $y_0 \in X, y(\cdot)$ is continuously differentiable and we have

$$y'(t) = AS(t)y_0 + C(t)y_1 + \int_0^t C(t-s)h(s)ds.$$

For additional details about cosine function theory, we refer the reader to [35, 36].

4.1. Existence of mild solutions

Now we give our main existence result for problem (3)-(4). Before starting and proving this result, we give the definition of a mild solution.

Definition 4.3. We say that a continuous function $y : (-\infty, +\infty) \rightarrow E$ is a mild solution of problem (3)-(4) if $y(t) = \phi(t), t \in (-\infty, 0], y'(0) = \varphi$ and

$$\begin{aligned} y(t) &= C(t)\phi(0) + S(t)[\varphi - g(0, \phi)] \\ &+ \int_0^t C(t-s)g(s, y_{\rho(s, y_s)})ds + \int_0^t S(t-s)f(s, y_{\rho(s, y_s)})ds, t \in J. \end{aligned} \quad (8)$$

Let us introduce the following hypothesis:

(H) $A : D(A) \subset E \rightarrow E$ is the infinitesimal generator of a uniformly continuous cosine function $(C(t))_{t \geq 0}$. Let

$$M_C = \sup\{\|C(t)\|_{B(E)} : t \geq 0\}, \quad M' = \sup\{\|S(t)\|_{B(E)} : t \geq 0\}.$$

Theorem 4.4. *Assume that (H), $(H_2) - (H_6)$, (H_ϕ) hold. If*

$$l(k^*M' + Mk_g^*) < 1, \quad (9)$$

then the problem (3)-(4) has at least one mild solution on BC .

Proof. We transform the problem (3)-(4) into a fixed point problem. Consider the operator: $N : BC \rightarrow BC$ define by:

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ C(t)\phi(0) + S(t)[\varphi - g(0, \phi)] \\ + \int_0^t C(t-s) g(s, y_{\rho(s, y_s)}) ds + \int_0^t S(t-s) f(s, y_{\rho(s, y_s)}) ds, & \text{if } t \in J. \end{cases}$$

Let $x(\cdot) : (-\infty, +\infty) \rightarrow E$ be the function defined by:

$$x(t) = \begin{cases} \phi(t); & \text{if } t \in (-\infty, 0]; \\ C(t)\phi(0); & \text{if } t \in J, \end{cases}$$

then $x_0 = \phi$. For each $z \in BC$ with $z(0) = 0$, $y'(0) = \varphi = z'(0) = \varphi_1$, we denote by \bar{z} the function

$$\bar{z}(t) = \begin{cases} 0; & \text{if } t \in (-\infty, 0]; \\ z(t); & \text{if } t \in J. \end{cases}$$

If y satisfies (8), we can decompose it as $y(t) = z(t) + x(t)$, $t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$ and the function $z(\cdot)$ satisfies

$$\begin{aligned} z(t) &= S(t)[\varphi_1 - g(0, \phi)] + \int_0^t C(t-s) g(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}) ds \\ &+ \int_0^t S(t-s) f(s, z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}) ds, \quad t \in J. \end{aligned}$$

Set

$$BC'_0 = \{z \in BC' : z(0) = 0\}$$

and let

$$\|z\|_{BC'_0} = \sup\{|z(t)| : t \in J\}, \quad z \in BC'_0.$$

BC'_0 is a Banach space with the norm $\|\cdot\|_{BC'_0}$. We define the operator $\mathcal{A} : BC'_0 \rightarrow BC'_0$ by:

$$\begin{aligned} \mathcal{A}(z)(t) &= S(t)[\varphi_1 - g(0, \phi)] + \int_0^t C(t-s) g(s, z_{\rho(s, z_s+x_s)} + x_{\rho(s, z_s+x_s)}) ds \\ &+ \int_0^t S(t-s) f(s, z_{\rho(s, z_s+x_s)} + x_{\rho(s, z_s+x_s)}) ds, \quad t \in J. \end{aligned}$$

We shall show that the operator \mathcal{A} satisfies all conditions of Mönch's fixed point theorem. The operator \mathcal{A} maps BC'_0 into BC'_0 , indeed the map $\mathcal{A}(z)$ is continuous on $[0, +\infty)$ for any $z \in BC'_0$, and for each $t \in J$ we have

$$\begin{aligned} |\mathcal{A}(z)(t)| &\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] \\ &+ M' \int_0^t |f(s, z_{\rho(s, z_s+x_s)} + x_{\rho(s, z_s+x_s)})| ds \\ &+ M \int_0^t |g(s, z_{\rho(s, z_s+x_s)} + x_{\rho(s, z_s+x_s)})| ds \\ &\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] \\ &+ M' \int_0^t k(s) \|z_{\rho(s, z_s+x_s)} + x_{\rho(s, z_s+x_s)}\|_{\mathcal{B}} ds \\ &+ M \int_0^t k_g(s) \|z_{\rho(s, z_s+x_s)} + x_{\rho(s, z_s+x_s)}\|_{\mathcal{B}} ds \\ &\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] \\ &+ M' \int_0^t k(s) (l|z(s)| + (m + \mathcal{L}^\phi + lMH)) \|\phi\|_{\mathcal{B}} ds \\ &+ M \int_0^t k_g(s) (l|z(s)| + (m + \mathcal{L}^\phi + lMH)) \|\phi\|_{\mathcal{B}} ds. \end{aligned}$$

Let

$$C = (m + \mathcal{L}^\phi + lMH) \|\phi\|_{\mathcal{B}}.$$

Then, we have:

$$\begin{aligned} |\mathcal{A}(z)(t)| &\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] \\ &+ M'C \int_0^t k(s) ds + M'l \int_0^t k(s) |z(s)| ds \\ &+ MC \int_0^t k_g(s) ds + Ml \int_0^t k_g(s) |z(s)| ds \\ &\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] + M' C k^* + M'l \|z\|_{BC'_0} k^* \\ &+ M C k_g^* + Ml \|z\|_{BC'_0} k_g^*. \end{aligned}$$

Set

$$C_1 = M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] + M' C k^* + M C k_g^*.$$

Hence, $\mathcal{A}(z) \in BC'_0$.

Moreover, let $r > 0$ be such that $r \geq \frac{C_1}{1-l(M'lk^*+Mk_g^*)}$, and B_r be the closed ball in BC'_0 centered at the origin and of radius r . Let $y \in B_r$ and $t \in [0, +\infty)$. Then,

$$|\mathcal{A}(z)(t)| \leq C_1 + M'lk^*r + Mlk_g^*r.$$

Thus,

$$\|\mathcal{A}(z)\|_{BC'_0} \leq r,$$

which means that the operator \mathcal{A} transforms the ball B_r into itself.

Now we prove that $\mathcal{A} : B_r \rightarrow B_r$ satisfies the assumptions of Mönch's fixed theorem. The proof will be given in several steps.

Step 1: \mathcal{A} is continuous in B_r .

Let $\{z_n\}$ be a sequence such that $z_n \rightarrow z$ in B_r . At the first, we study the convergence of the sequences $(z_{\rho(s, z_s^n)}^n)_{n \in \mathbb{N}, s \in J}$.

If $s \in J$ is such that $\rho(s, z_s) > 0$, then we have,

$$\begin{aligned} \|z_{\rho(s, z_s^n)}^n - z_{\rho(s, z_s)}\|_{\mathcal{B}} &\leq \|z_{\rho(s, z_s^n)}^n - z_{\rho(s, z_s^n)}\|_{\mathcal{B}} + \|z_{\rho(s, z_s^n)} - z_{\rho(s, z_s)}\|_{\mathcal{B}} \\ &\leq l\|z_n - z\|_{B_r} + \|z_{\rho(s, z_s^n)} - z_{\rho(s, z_s)}\|_{\mathcal{B}}, \end{aligned}$$

which proves that $z_{\rho(s, z_s^n)}^n \rightarrow z_{\rho(s, z_s)}$ in \mathcal{B} as $n \rightarrow \infty$ for every $s \in J$ such that $\rho(s, z_s) > 0$. Similarly, is $\rho(s, z_s) < 0$, we get

$$\|z_{\rho(s, z_s^n)}^n - z_{\rho(s, z_s)}\|_{\mathcal{B}} = \|\phi_{\rho(s, z_s^n)}^n - \phi_{\rho(s, z_s)}\|_{\mathcal{B}} = 0$$

which also shows that $z_{\rho(s, z_s^n)}^n \rightarrow z_{\rho(s, z_s)}$ in \mathcal{B} as $n \rightarrow \infty$ for every $s \in J$ such that $\rho(s, z_s) < 0$. Combining the pervious arguments, we can prove that $z_{\rho(s, z_s)}^n \rightarrow \phi$ for every $s \in J$ such that $\rho(s, z_s) = 0$. Finally,

$$\begin{aligned} &|\mathcal{A}(z_n)(t) - \mathcal{A}(z)(t)| \\ &\leq M' \int_0^t |f(s, z_{\rho(s, z_s^n)}^n + x_{\rho(s, z_s^n)} + x_s) - f(s, z_{\rho(s, z_s)} + x_{\rho(s, z_s)} + x_s)| ds \\ &+ M \int_0^t |g(s, z_{\rho(s, z_s^n)}^n + x_{\rho(s, z_s^n)} + x_s) - g(s, z_{\rho(s, z_s)} + x_{\rho(s, z_s)} + x_s)| ds. \end{aligned}$$

Then by (H_2) , (H_5) we have

$$f(s, z_{\rho(s, z_s^n)}^n + x_{\rho(s, z_s^n)} + x_s) \rightarrow f(s, z_{\rho(s, z_s)} + x_{\rho(s, z_s)} + x_s), \text{ as } n \rightarrow \infty,$$

$$g(s, z_{\rho(s, z_s^n)}^n + x_{\rho(s, z_s^n)} + x_s) \rightarrow g(s, z_{\rho(s, z_s)} + x_{\rho(s, z_s)} + x_s), \text{ as } n \rightarrow \infty,$$

and by the Lebesgue dominated convergence theorem we get,

$$\|\mathcal{A}(z_n) - \mathcal{A}(z)\|_{BC'_0} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Thus \mathcal{A} is continuous.

Step 2 : $\mathcal{A}(B_r) \subset B_r$ this is clear.

Step 3: $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0, b]$ of $[0, +\infty)$ for $b > 0$. Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, we have

$$\begin{aligned}
& |\mathcal{A}(z)(\tau_2) - \mathcal{A}(z)(\tau_1)| \\
& \leq \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)} [\|\varphi_1\| - g(0, \phi)] \\
& + \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)} |f(s, z_{\rho(s, z_s^n)}^n + x_{\rho(s, z_s^n + x_s)})| ds \\
& + \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)} |f(s, z_{\rho(s, z_s^n)}^n + x_{\rho(s, z_s^n + x_s)})| ds \\
& + \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} |g(s, z_{\rho(s, z_s^n)}^n + x_{\rho(s, z_s^n + x_s)})| ds \\
& + \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} |g(s, z_{\rho(s, z_s^n)}^n + x_{\rho(s, z_s^n + x_s)})| ds \\
& \leq \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)} [\|\varphi_1\| - g(0, \phi)] \\
& + C \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)} k(s) ds \\
& + lr \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)} k(s) ds \\
& + C \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)} k(s) ds \\
& + lr \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)} k(s) ds \\
& + C \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} k_g(s) ds \\
& + lr \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} k_g(s) ds \\
& + C \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} k_g(s) ds \\
& + lr \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} k_g(s) ds.
\end{aligned}$$

When $\tau_1 \rightarrow \tau_2$, the right-hand side of the above inequality tends to zero, since $C(t), S(t)$ are a uniformly continuous operator (see [35, 36]). This proves the equicontinuity.

Step 4: $\mathcal{A}(B_r)$ is equiconvergent.

Let $y \in B_r$, we have:

$$\begin{aligned} |\mathcal{A}(z)(t)| &\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}}] + M' \int_0^t |f(s, z_{\rho(s, z_s^n + x_s)}^n + x_{\rho(s, z_s^n + x_s)})| ds \\ &+ M \int_0^t |g(s, z_{\rho(s, z_s^n + x_s)}^n + x_{\rho(s, z_s^n + x_s)})| ds \\ &\leq C_1 + M'rl \int_0^t k(s) ds + Mrl \int_0^t k_g(s) ds. \end{aligned}$$

Then

$$\lim_{t \rightarrow +\infty} |\mathcal{A}(z)(t)| \leq C_2,$$

where

$$C_2 \leq C_1 + rl(M'k^* + Mk_g^*).$$

Hence,

$$|\mathcal{A}(z)(t) - \mathcal{A}(z)(+\infty)| \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

Now let V be a subset of B_r such that $V \subset \overline{\text{conv}}(\mathcal{A}(V) \cup \{0\})$. V is bounded and equicontinuous and therefore the function $t \rightarrow v(t) = \alpha(V(t))$ is continuous on \mathbb{R} .

$$\begin{aligned} V(t) &\leq \alpha(\mathcal{A}(V)(t) \cup \{0\}) \leq \alpha(\mathcal{A}(V)(t)) \\ &\leq M \int_0^t k_g(s) \alpha(V(s)) ds + M' \int_0^t k(s) \alpha(V(s)) ds \\ &\leq M \int_0^t k_g(s) v(s) ds + M' \int_0^t k(s) v(s) ds \\ &\leq l(Mk_g^* + M'k^*) \|v\|_{\infty}. \end{aligned}$$

This means that

$$\|v\|_{\infty} (1 - l(k_g^*M + M'k^*)) \leq 0$$

By (9) it follows that $\|v\|_{\infty} = 0$, that is $v(t) = 0$ for each $t \in J$ and then $V(t)$ is relatively compact in E . From Mönch's theorem, we deduce that \mathcal{A} has a fixed point z^* . Then $y^* = z^* + x$ is a fixed point of the operators N , which is a mild solution of the problem (3)-(4). \square

5. Examples

5.1. Example 1

Consider the following neutral functional partial differential equation:

$$\frac{\partial}{\partial t} [z(t, x) - g(t, z(t - \sigma(t, z(t, 0)), x))] = \frac{\partial^2}{\partial x^2} [z(t, x) - g(t, z(t - \sigma(t, z(t, 0)), x))]$$

$$+f(t, z(t - \sigma(t, z(t, 0)), x)), \quad x \in [0, \pi], \quad t \in [0, +\infty) \quad (10)$$

$$z(t, 0) = z(t, \pi) = 0, \quad t \in [0, +\infty), \quad (11)$$

$$z(\theta, x) = z_0(\theta, x), \quad t \in (-\infty, 0], \quad x \in [0, \pi], \quad (12)$$

where f, g are given functions, and $\sigma : \mathbb{R} \rightarrow \mathbb{R}^+$. Take $E = L^2[0, \pi]$ and define $A : E \rightarrow E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E, \omega, \omega' \text{ are absolutely continuous, } \omega'' \in E, \omega(0) = \omega(\pi) = 0\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2 (\omega, \omega_n) \omega_n, \quad \omega \in D(A),$$

where $\omega_n(s) = \sqrt{\frac{2}{\pi}} \sin ns, n = 1, 2, \dots$ is the orthogonal set of eigenvectors in A . It is well known (see [31]) that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t) (\omega, \omega_n) \omega_n, \quad \omega \in E.$$

Since the analytic semigroup $T(t)$ is compact for $t > 0$, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BCU(\mathbb{R}^-; E)$ the space of uniformly continuous and bounded functions from \mathbb{R}^- into E and $\phi \in \mathcal{B}$, then (H_ϕ) , where $\rho(t, \varphi) = t - \sigma(\varphi)$.

Hence, the problem (1)-(2) in an abstract formulation of the problem (10)-(12), and if the conditions $(H_1) - (H_7)$, (H_ϕ) are satisfied. Theorem 3.4 implies that the problem (10)-(12) has at least one mild solutions on BC .

5.2. Example 2

Take $E = L^2[0, \pi]; \mathcal{B} = C_0 \times L^2(h, E)$ and define $A : E \rightarrow E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E; \omega, \omega' \text{ are absolutely continuous, } \omega'' \in E, \omega(0) = \omega(\pi) = 0\}.$$

It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t \in \mathbb{R}}$ on E , respectively. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) := \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin n\tau,$$

and the following properties hold:

- (a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of E .
- (b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 \langle y, z_n \rangle z_n$.
- (c) For $y \in E$, $C(t)y = \sum_{n=1}^{\infty} \cos(nt) \langle y, z_n \rangle z_n$, and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle y, z_n \rangle z_n$$

which implies that the operator $S(t)$ is compact, for all $t \in J$ and that

$$\|C(t)\| = \|S(t)\| \leq 1, \text{ for all } t \in \mathbb{R}.$$

- (d) If Φ denotes the group of translations on E defined by

$$\Phi(t)y(\xi) = \tilde{y}(\xi + t),$$

where \tilde{y} is the extension of y with period 2π . Then

$$C(t) = \frac{1}{2} (\Phi(t) + \Phi(-t)), \quad A = B^2,$$

where B is the infinitesimal generator of the group Φ on

$$X = \{y \in H^1(0, \pi) : y(0) = y(\pi) = 0\}.$$

Consider the functional partial differential equation of second order:

$$\begin{aligned} \frac{\partial}{\partial t} \left[\frac{\partial}{\partial t} z(t, x) + \int_{-\infty}^0 b(s-t) z(s - \rho_1(t) \rho_2(|z(t)|), x) ds \right] &= \frac{\partial^2}{\partial x^2} z(t, x) \\ &+ \int_{-\infty}^0 a(s-t) z(s - \rho_1(t) \rho_2(|z(t)|), x) ds, \end{aligned}$$

$$x \in [0, \pi], \quad t \in J := [0, +\infty), \quad (13)$$

$$z(t, 0) = z(t, \pi) = 0, \quad t \in [0, +\infty), \quad (14)$$

$$z(t, x) = \phi(t, x), \quad \frac{\partial z(0, x)}{\partial t} = \varphi(x), \quad t \in [-r, 0], \quad x \in [0, \pi], \quad (15)$$

where $\phi \in \mathcal{B}$, $\rho_i : [0, \infty) \rightarrow [0, \infty)$, $a, b : \mathbb{R} \rightarrow \mathbb{R}$ be continuous, and

$$L_f = \int_{-\infty}^0 \frac{a^2(s)}{2h(s)} ds < \infty, \quad L_g = \int_{-\infty}^0 \frac{b^2(s)}{2h(s)} ds < \infty.$$

Under these conditions, we define the functions $f : J \times \mathcal{B} \rightarrow E$, $\rho : J \times \mathcal{B} \rightarrow \mathbb{R}$ by

$$f(t, \psi)(x) = \int_{-\infty}^0 a(s) \psi(s, x) ds,$$

$$g(t, \psi)(x) = \int_{-\infty}^0 b(s)\psi(s, x)ds,$$

$$\rho(s, \psi) = s - \rho_1(s)\rho_2(|\psi(0)|),$$

we have

$$\|f(t, \cdot)\|_{\mathfrak{B}(\mathcal{B}, E)} \leq L_f, \text{ and } \|g(t, \cdot)\|_{\mathfrak{B}(\mathcal{B}, E)} \leq L_g.$$

Then the problem (3)-(4) in an abstract formulation of the problem (13)-(15). If conditions $(H_2) - (H_6), (H_\phi)$ are satisfied, Theorem 4.4 implies that the problem (13)-(15) has at least one mild solution on BC .

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On the Evolution of Academic Staff Structure in a University Setting

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ABSTRACT: This paper models the academic staff structure in a university as a system of stocks and flows in a three-dimensional space, \mathbb{R}^3 . The stocks are the number of academic staff in a particular state at a given time and the flows are the staff moving between any two states over an interval of time. The paper places emphasis on the grade-specific completion rates of Graduate Assistants, who choose to study in the university in which they are employed for higher degrees. The study describes the evolution of structures in the university as a linear recurrence system. Some aspects of linear algebra are employed as a theoretical underpinning to gain insights into the transformation matrix of the recurrence system. A number of resulting propositions are presented along with their proofs. We provide two theorems to serve as a means of predicting a university manpower structure. Following that a numerical illustration of the theorems and propositions is provided with data which are representative of the kind of data in a Nigerian university system.

AMS Subject Classification: 15A18, 91D35.

Keywords and Phrases: Diagonalisable matrices; Eigenvalues; Linear mapping; Manpower planning; Recurrence system.

1. Introduction

The evolution of staff structure in a manpower system is a key issue attracting the attention of management, planners and policy makers, particularly in a university system. This is because the future distribution of staff could aid management in planning and resource allocation, which may include framing policies to guide against shortages and excess staffing requirements and budgeting for personnel costs and staff development programmes. The population of academic staff in a university is viewed as a system of interconnected states in such fashion that each individual academic staff can follow a path to become a professor. In this light a university manpower

system may be called a hierarchical system. Authors such as Vassiliou and Tsantas [17], Guerry [6] and Kipouridis and Tsaklidis [8] analyzed hierarchical manpower systems using discrete-time Markov chains. The semi-Markov models [14, 16] and the continuous-time Markov models [5, 10] have also found applications in hierarchical manpower systems. More details on the use of manpower planning models are found in Bartholomew et al.[1]. Mathematical models for predicting the future manpower needs in the education sector were reviewed by Johnes [7]. However, the models in [7] do not take into consideration the grade-specific supervision completion rates of temporary staff (Graduate Assistants in this study), who choose to study in the institution in which they are employed for higher qualification(s) with a view to enhancing their career prospects. It is against this background we attempt to revisit manpower forecasting by taking into consideration the grade-specific supervision completion rates. It is important to mention here that predicting the future distribution of staff in a university system is an uphill task as the flows of staff in the system cannot entirely be controlled. One of such flows is wastage which may be due to resignation, death, ill-health, etc. Hence, most hierarchical models in the literature have adopted the use of mathematical expectation [1].

This study focuses on a university manpower system with a countable state space $S, 0 \notin S$. The two-way flow between the system and the outside world, due to recruitment and wastage, is introduced by constructing a sample space $\Omega = 0 \cup S$ with 0 being a state external to the system. The total number of transitions in Ω is $(\#(\Omega))^2$, where $\#$ is used to denote the cardinality of a set. Nonetheless, we place a constraint on the transition process by assuming a scenario where the transitions follow a natural order, that is, from one grade to either itself or the next higher grade without demotion or double promotion. This constrain is representative of the kind of data in a Nigerian university setting. We view the manpower system of the university as a random process made up of stocks and flows in a three-dimensional space \mathbb{R}^3 according to $\#(S) = 3$. This is in line with the staff-mix by rank of the National Universities Commission [3, 13]. Considering a grade-specific supervision completion rates for the Graduate Assistants to obtain a higher degree (usually, a master's degree) from the university of employment within a specific time bound and that recruitment is done to replace leavers and to achieve the desired growth [4, 15], we develop a recurrence system where a linear mapping $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is capable of predicting the future academic staff structure. The basic properties of the transformation matrix of the linear mapping are given with proofs. The transformation matrix has a stable set of parameters.

1.1. A brief description

It is rather useful to write something about the university system, which suits the Nigerian university setting considered in this paper. Academic staff in the university system is stratified into three states: the professorial cadre, which is made up of Professors and Associate Professors; the cadre of senior lecturers; and the junior academic staff cadre comprising the rank of Lecturer I and below, excluding the position of Graduate Assistant. This kind of stratification is being adopted by the National

Universities Commission [13]. Thus academic staff may be listed from the rank of Assistant Lecturer to the rank of Professor. The rank may be used as a determinant of a staff commitment [11]. The position of Graduate Assistant in Nigerian universities is a temporary appointment for academic staff without a master's degree. The terms and conditions for this appointment have a clause which stipulates a maximum time bound (usually four calendar years) for the staff to obtain a higher degree, otherwise such a staff loses the position. A Graduate Assistant (GA) may either enrol for a higher degree programme in the university of employment or apply for a leave of absence to study elsewhere. In either case, the maximum time bound must not be exceeded. In the first case, the tuition fee is borne by the university and the staff are under the tutelage of the members of the Postgraduate Board of Studies and Examiners. In most university settings, the task of postgraduate (PG) supervision is limited to staff in the rank of Senior Lecturer and the professorial cadre. Thus, a member of the PG board contributes a certain number of assistant lecturers to the junior academic staff cadre. The size of the PG board has great implications on the number of supervisors and candidates admitted into higher degree programmes [12]. Each member of academic staff holds a membership, which is given up when the staff leaves the university (due to resignation, retirement, retrenchment, etc.) and is taken by new recruits into the system. The progression of academic staff through the ranks in the university is normally by promotion and upgrading. Upgrading applies to staff who possesses higher qualification(s) to move to the next higher rank without necessarily completing the length of service interval for promotion, while promotion is a move to the next higher rank after satisfying certain requirements and completing the length of service interval.

1.2. Basic definitions

To make this study clearer to a broader audience, we define some key terms and notations. The definitions are credited to [2]. Except where otherwise stated, this study assumes that \mathbf{A} is an $n \times n$ square matrix.

Definition 1.2.1. The set $sp(\mathbf{A}) = \{\lambda \in \mathbf{C} : det(\mathbf{A} - \lambda\mathbf{I}) = 0\}$ forms the set of eigenvalues of matrix \mathbf{A} and $sp(\mathbf{A})$ is called the spectrum of matrix \mathbf{A} .

Definition 1.2.2. The real number $\rho(\mathbf{A}) = \max\{|\lambda| : \lambda \in sp(\mathbf{A})\}$ is called the spectral radius of matrix \mathbf{A} .

Definition 1.2.3. An eigenvalue of algebraic multiplicity unity is called simple; otherwise it is said to be multiple.

Definition 1.2.4. An eigenvalue of multiplicity m greater than unity is said to be semi-simple if it admits m linearly independent eigenvectors $\mathbf{0} \neq \mathbf{X} \in \mathbf{C}^n$ such that $\mathbf{A}\mathbf{X} = \lambda\mathbf{X}$; otherwise it is said to be defective.

Definition 1.2.5. A diagonalisable matrix is a matrix whose eigenvalues are semi-simple.

Definition 1.2.6. A matrix is said to be non-negative if the elements are either positive or zero.

2. Methodology

This section contains theorems on the recurrence system for the academic staff structure, the basic properties of the transformation matrix and some remarks.

2.1. Model development

We consider a university system with a stable manpower expansion rate. Let the evolution of academic staff structure for the university be required at the end of a period of length Δ years from an interval of the form $[0, \Delta), [\Delta, 2\Delta), \dots, [(\beta - 1)\Delta, \beta\Delta)$, $\beta \geq 1$. More specifically, we choose Δ to coincide with the maximum length of stay for GA's. In most universities in Nigeria such as the University of Benin, $\Delta = 4$. Let t be a measure of time in a discrete time scale of a multiple of Δ years. Let $S = \{1, 2, 3\}$ be a state space for the university system, where state 1 stands for the junior academic staff cadre, state 2 for the cadre for senior lecturers and state 3 for the professorial cadre. The states are assumed to be mutually exclusive and exhaustive. Let $\mathbf{n}(t) \in \mathbb{R}^3$ be a column vector with entries denoted as $n_i(t)$ being the number of staff in state $i \in S$ at time t . The number $n_i(t)$ is the manpower stock in state i . The vector $\mathbf{n}(t)$ is called the academic staff structure (or grade structure) at time t . The initial academic staff structure is assumed known (that is, no 'ghost' worker) and it is denoted as $\mathbf{n}(0)$. The realisation of the system at time t is the total number of staff given as $N(t) = \mathbf{e}\mathbf{n}(t)$, where $\mathbf{e} \in \mathbb{R}^3$ is a row vector of ones. The flow of staff from a state i to a state $j = i, i + 1 \in S$, in a given period t is assumed to be a random variable with a binomial distribution and the parameters of the distribution are the stock in state i and the transition probability from state i to state j in period t .

Let $p_{ij}(t)$ be the transition probability from state i to j in period $t, i, j \in S$. We assume a system with a stable set of parameters and policies so that $p_{ij}(t) = p_{ij}$ for all t . This assumption was earlier employed by Nilakantan [11]. The transition probabilities satisfy the relation $p_{i1} + p_{i2} + p_{i3} \leq 1$ for each $i \in S$, owing to the loss of staff. Let m_j denote a grade-specific supervision completion rate for GA's assigned to a supervisor in state $j > 1$ in the interval $[t, t + \Delta]$. The m_j 's are assumed constant in time. Since PG supervision is restricted to members of the PG board, then $m_1 = 0$ and $m_2, m_3 \in \mathbb{R}$. The number of GA's upgraded to the rank of Assistant Lecturer in an interval of length Δ , which in turn expands the junior academic staff cadre, is given by $m_j n_j(t)$. We treat GA's who have obtained a higher degree outside their university of employment as a part of new recruits into the junior academic staff cadre. Let p_{0j} denote the fraction of recruits into category $j \in S$, where $0 \notin S$ is a state where staff who leave the system are transferred. Let g denote the desired expansion rate for the system. We assume that recruitment to reach the desired expansion rate is done independently of internal movements. By the assumption that recruitment is done to replace wastage and to achieve the desired expansion [15], the number of recruits into a state i in the next period is obtained as

$$\sum_{i=1}^3 (1 - p_{ii} - p_{ij}) n_i(t) + g \left(\sum_{i=1}^3 n_i(t) \right), \quad i, j = i + 1 \in S, \quad p_{34} = 0.$$

With the aforementioned assumptions and notations at hand, we give the main theorem to provide a tool for practical applications.

Theorem 2.1.1. *Let there be a university system, where the academic staff are stratified into three states and temporary recruits as Graduate Assistants may choose to study in the university in which they are employed. Assuming a stable set of parameters and policies and that recruitment is done to replace wastage and to achieve the desired expansion, the evolution of staff structures in the university is a linear recurrence system of the form*

$$\mathbf{n}(t + \Delta) = \mathbf{A}\mathbf{n}(t), \quad t = 0, \Delta, 2\Delta, \dots,$$

up to the latest time we wish to project to, with $\mathbf{A} = \mathbf{Q}' \in \mathbb{R}^{3 \times 3}$, where the entries q_{ij} in \mathbf{Q} are defined by

$$q_{ij} = \begin{cases} \phi_{ij} + p_{0j}(1 - p_{ii} - p_{i,i+1} + g) & \text{for } i \neq 3 \\ \phi_{ij} + p_{0j}(1 - p_{ii} + g) & \text{for } i = 3, \end{cases}$$

$\phi_{ij} = p_{ij}$ for $j = i, i + 1$, $\phi_{i1} = m_i$ for $i > 1$, $\phi_{13} = \phi_{32} = 0$, and the prime ' denotes a matrix transpose.

Proof. Consider the accounting equation for each state i in S of the system:

$$n_1(t + \Delta) = \sum_{i=1}^3 \phi_{i1}n_i(t) + p_{01} \left(\sum_{i=1}^3 (1 - p_{ii} - p_{ij})n_i(t) + g \left(\sum_{i=1}^3 n_i(t) \right) \right),$$

$$n_2(t + \Delta) = p_{12}n_1(t) + p_{22}n_2(t) + p_{02} \left(\sum_{i=1}^3 (1 - p_{ii} - p_{ij})n_i(t) + g \left(\sum_{i=1}^3 n_i(t) \right) \right),$$

and

$$n_3(t + \Delta) = p_{23}n_2(t) + p_{33}n_3(t) + p_{03} \left(\sum_{i=1}^3 (1 - p_{ii} - p_{ij})n_i(t) + g \left(\sum_{i=1}^3 n_i(t) \right) \right).$$

Rewriting the accounting equations as a single matrix equation, we get

$$\mathbf{n}(t + \Delta) = \mathbf{L}'\mathbf{n}(t) + \mathbf{P}'_0 (\mathbf{e} [\mathbf{I} - \mathbf{P}'] \mathbf{n}(t) + g\mathbf{en}(t)),$$

where

$$\mathbf{L}' = \begin{bmatrix} p_{11} & m_2 & m_3 \\ p_{12} & p_{22} & 0 \\ 0 & p_{23} & p_{33} \end{bmatrix} \text{ is a } 3 \times 3 \text{ Leslie matrix, } \mathbf{P}'_0 = \begin{bmatrix} p_{01} \\ p_{02} \\ p_{03} \end{bmatrix} \text{ is a } 3 \times 1$$

$$\text{recruitment vector, } \mathbf{P}' = \begin{bmatrix} p_{11} & 0 & 0 \\ p_{12} & p_{22} & 0 \\ 0 & p_{23} & p_{33} \end{bmatrix} \text{ is a } 3 \times 3 \text{ sub-stochastic transition}$$

matrix, $\mathbf{e} = [1 \ 1 \ 1]$ and

$$\mathbf{I} = [\delta_{ij}] \text{ with } \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j, \end{cases} \quad i, j \in S.$$

The vector $\mathbf{L}'\mathbf{n}(t) + \mathbf{P}'_0(\mathbf{e}[\mathbf{I} - \mathbf{P}']\mathbf{n}(t) + g\mathbf{en}(t))$ simplifies to $(\mathbf{L}' + \mathbf{P}'_0\mathbf{e}[(1+g)\mathbf{I} - \mathbf{P}'])\mathbf{n}(t)$, where the matrix $(\mathbf{L}' + \mathbf{P}'_0\mathbf{e}[(1+g)\mathbf{I} - \mathbf{P}']) = \mathbf{Q}'$. This completes the proof. \square

2.2. Basic properties

Theorem 2.1.1 above provides a recurrence framework to describe the academic staff structure of a university system. The behaviour of the system over time depends on the nature of matrix \mathbf{A} . Clearly, matrix \mathbf{A} is a non-negative matrix. Nonetheless, there are other interesting properties of matrix \mathbf{A} . We state these properties as propositions with proofs.

Proposition 2.2.1. The eigenvalue $\lambda \in sp(\mathbf{A}) \subset \mathbf{C}$ is given by

$$\lambda = \begin{cases} \frac{1}{3}tr(\mathbf{A}) + \left(\det(\mathbf{A}) - \frac{1}{27}tr^3(\mathbf{A})\right)^{\frac{1}{3}} \exp\left(z\left(\frac{2}{3}(k+1)\pi\right)\right) & \text{if } tr^2(\mathbf{A}) = 3S_2 \\ \frac{1}{3}tr(\mathbf{A}) + r \cos\left(\frac{2k\pi}{3} + \frac{1}{3} \arccos \zeta\right) & \text{if } tr^2(\mathbf{A}) \neq 3S_2, \end{cases}$$

where $k = 0, 1, 2$, $z = \sqrt{-1}$, $tr(\mathbf{A})$ is the trace of matrix \mathbf{A} , S_2 is the sum of the principal minors of matrix \mathbf{A} of order 2, $r = \frac{2}{3}(tr^2(\mathbf{A}) - 3S_2)^{\frac{1}{2}}$, and

$$\zeta = \frac{27\left(\det(\mathbf{A}) - \left(\frac{1}{3}tr(\mathbf{A})\right)^3\right) + 3tr(\mathbf{A})(tr^2(\mathbf{A}) - 3S_2)}{2(tr^2(\mathbf{A}) - 3S_2)^{\frac{3}{2}}}.$$

Proof. Let $P(\lambda)$ denote the characteristic polynomial of matrix \mathbf{A} , which is expressed as [9]

$$P(\lambda) = \lambda^3 - tr(\mathbf{A})\lambda^2 + S_2\lambda - \det(\mathbf{A}).$$

Since the eigenvalues $\lambda \in sp(\mathbf{A})$ are obtained by solving the singular pencil $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, we set

$$\lambda^3 - tr(\mathbf{A})\lambda^2 + S_2\lambda - \det(\mathbf{A}) = 0.$$

Let $\lambda = x + \frac{1}{3}tr(\mathbf{A})$. Then

$$x^3 + \left(S_2 - \frac{1}{3}tr^2(\mathbf{A})\right)x + \frac{1}{3}tr(\mathbf{A})\left(S_2 - \frac{2}{9}tr^2(\mathbf{A})\right) - \det(\mathbf{A}) = 0.$$

If $tr^2(\mathbf{A}) = 3S_2$, then

$$x = \left(\det(\mathbf{A}) - \frac{1}{27}tr^3(\mathbf{A})\right)^{\frac{1}{3}}, \left(\det(\mathbf{A}) - \frac{1}{27}tr^3(\mathbf{A})\right)^{\frac{1}{3}} \left(\cos \frac{2\pi}{3} \pm z \sin \frac{2\pi}{3}\right).$$

Using the exponential form of a complex number, we obtain

$$\lambda = \frac{1}{3}tr(\mathbf{A}) + \left(det(\mathbf{A}) - \frac{1}{27}tr^3(\mathbf{A}) \right)^{\frac{1}{3}} \exp \left(z \left(\frac{2}{3}(k+1)\pi \right) \right), \quad k = 0, 1, 2.$$

On the other hand, if $tr^2(\mathbf{A}) \neq 3S_2$, then we set $x = r \cos \theta$ and then compare the resulting equation

$$r^3 \cos^3 \theta + \left(S_2 - \frac{1}{3}tr^2(\mathbf{A}) \right) r \cos \theta = det(\mathbf{A}) - \frac{1}{3}tr(\mathbf{A}) \left(S_2 - \frac{2}{9}tr^2(\mathbf{A}) \right)$$

with the trigonometric identity

$$4 \cos^3 \theta - 3 \cos \theta = \cos 3\theta.$$

Thus,

$$\frac{r^3}{4} = \frac{\left(\frac{1}{3}tr^2(\mathbf{A}) - S_2 \right) r}{3} = \frac{det(\mathbf{A}) - \frac{1}{3}tr(\mathbf{A}) \left(S_2 - \frac{2}{9}tr^2(\mathbf{A}) \right)}{\cos 3\theta}.$$

Further simplifications yield

$$r = 0, \quad \pm \frac{2}{3} \left(tr^2(\mathbf{A}) - 3S_2 \right)^{\frac{1}{2}}, \text{ and}$$

$$\cos 3\theta = \frac{3 \left(det(\mathbf{A}) - \left(\frac{1}{27}tr(\mathbf{A}) \right)^3 \right) + \frac{1}{3}tr(\mathbf{A}) \left(tr^2(\mathbf{A}) - 3S_2 \right)}{\left(\frac{1}{3}tr^2(\mathbf{A}) - S_2 \right) r}.$$

We exclude the case $r = 0$ as division by zero is undefined. With

$$r = \frac{2}{3} \left(tr^2(\mathbf{A}) - 3S_2 \right)^{\frac{1}{2}},$$

we find

$$\theta = \frac{2k\pi}{3} + \frac{1}{3} \arccos \left(\frac{27 \left(det(\mathbf{A}) - \left(\frac{1}{3}tr(\mathbf{A}) \right)^3 \right) + 3tr(\mathbf{A}) \left(tr^2(\mathbf{A}) - 3S_2 \right)}{2 \left(tr^2(\mathbf{A}) - 3S_2 \right)^{\frac{3}{2}}} \right),$$

$k = 0, 1, 2$. The case $r = -\frac{2}{3} \left(tr^2(\mathbf{A}) - 3S_2 \right)^{\frac{1}{2}}$ reproduces the same eigenvalues as with $r = \frac{2}{3} \left(tr^2(\mathbf{A}) - 3S_2 \right)^{\frac{1}{2}}$. \square

Proposition 2.2.2. For a non-contracting system, the trace $tr(\mathbf{A})$ lies in the interval $[0, 9)$.

Proof. By definition of the trace of a matrix,

$$tr(\mathbf{A}) = p_{11} + p_{01} (1 - p_{11} - p_{12} + g) + p_{22} + p_{02} (1 - p_{22} - p_{23} + g) \\ + p_{33} + p_{03} (1 - p_{33} + g).$$

Since $p_{ij}, i, j \in S$, is a transition probability, $0 \leq p_{ij} \leq 1$. Furthermore, $(1 - p_{ii} - p_{i,i+1})$ and p_{0j} are respectively wastage and recruitment probabilities; so they lie in the interval $[0, 1]$. For $0 \leq g < 1$, $tr(\mathbf{A}) \geq 0$ and $p_{ii} + p_{0j}(1 - p_{ii} - p_{i,i+1} + g) < 3$ for each $i = j \in S$. Hence $0 \leq tr(\mathbf{A}) < 9$. \square

Proposition 2.2.3. The spectral radius $\rho(\mathbf{A})$ satisfies the relation $\rho(\mathbf{A}) < \Psi$, where

$$\Psi = \begin{cases} 3 + \left| \det(\mathbf{A}) - \frac{1}{27}tr^3(\mathbf{A}) \right|^{\frac{1}{3}} & \text{if } tr^2(\mathbf{A}) = 3S_2 \\ 3 + |r| (\cosh^2 b - \sin^2 a)^{\frac{1}{2}} & \text{if } tr^2(\mathbf{A}) \neq 3S_2, \end{cases}$$

and the argument $\left(\frac{2k\pi}{3} + \frac{1}{3} \arccos \zeta\right)$ in Proposition 2.2.1 is of the form $a + zb \in \mathbf{C}$, $a, b \in \mathbf{R}$.

Proof. The proof follows from Propositions 2.2.1 and 2.2.2. If $tr^2(\mathbf{A}) = 3S_2$, then

$$\begin{aligned} |\lambda| &\leq \left| \frac{1}{3}tr(\mathbf{A}) \right| + \left| \left(\det(\mathbf{A}) - \frac{1}{27}tr^3(\mathbf{A}) \right)^{\frac{1}{3}} \exp \left(z \left(\frac{2}{3}(k+1)\pi \right) \right) \right| \\ &< 3 + \left| \det(\mathbf{A}) - \frac{1}{27}tr^3(\mathbf{A}) \right|^{\frac{1}{3}}. \end{aligned}$$

On the other hand, if $tr^2(\mathbf{A}) \neq 3S_2$, then

$$|\lambda| \leq \left| \frac{1}{3}tr(\mathbf{A}) \right| + \left| r \cos \left(\frac{2k\pi}{3} + \frac{1}{3} \arccos \zeta \right) \right|.$$

It follows that

$$|\lambda| < 3 + |r| |\cos a \cosh b - z \sin a \sinh b| \text{ as } \left(\frac{2k\pi}{3} + \frac{1}{3} \arccos \zeta \right) = a + zb.$$

Thus

$$|\lambda| < 3 + |r| (\cos^2 a \cosh^2 b + \sin^2 a \sinh^2 b)^{\frac{1}{2}} = 3 + |r| (\cosh^2 b - \sin^2 a)^{\frac{1}{2}}.$$

\square

Proposition 2.2.4. If $\det(\mathbf{A}) \neq \frac{1}{27}tr^3(\mathbf{A})$, then the transformation matrix \mathbf{A} is diagonalisable over the complex field \mathbf{C} .

Proof. If $\det(\mathbf{A}) \neq \frac{1}{27}tr^3(\mathbf{A})$, then, from Proposition 2.2.1, λ is simple as its algebraic multiplicity is unity. Since $\lambda \in \mathbf{C}$, matrix \mathbf{A} is diagonalisable over the complex field \mathbf{C} . \square

2.3. Some remarks

We make some remarks on the staff structure $\mathbf{n}(t + \Delta)$ and its representation.

Remark 2.3.1. *On examining the structure $\mathbf{n}(t + \Delta)$ as $t \rightarrow \infty$, Proposition 2.2.3 indicates that the system may be unstable and $\mathbf{n}(t + \Delta)$ may be unbounded.*

Remark 2.3.2. *Proposition 2.2.4 implies that the linear recurrence system given in Theorem 2.1.1 may be written as*

$$\mathbf{n}(0), \quad \mathbf{n}(\beta\Delta) = \mathbf{X}\mathbf{D}^\beta\mathbf{X}^{-1}\mathbf{n}(0), \quad \beta = 1, 2, 3, \dots,$$

according to $t = 0, \Delta, 2\Delta, \dots$, where $\mathbf{D} = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i \in \text{sp}(\mathbf{A}) \subset \mathbf{C}$, $i = 1, 2, 3$, and the i th column of \mathbf{X} is the right eigenvector \mathbf{X}_i associated with the eigenvalue λ_i .

Remark 2.3.2 enables us to make the following claim.

Theorem 2.3.1. *Let the transformation matrix, \mathbf{A} , be diagonalisable over the complex field, $\mathbf{C}^{3 \times 3}$. Then the manpower structure can be determined at any instant of time in the complex field.*

Proof. For $\lambda_i \in \text{sp}(\mathbf{A}) \subset \mathbf{C}$, $i \in S$, $\lambda_i^{\frac{\beta}{\alpha}}$ is defined in \mathbf{C} , for all $\alpha, \beta \in \mathbb{R}$. Hence \mathbf{X} , $\text{diag}\left(\lambda_1^{\frac{\beta}{\alpha}}, \lambda_2^{\frac{\beta}{\alpha}}, \lambda_3^{\frac{\beta}{\alpha}}\right)$, $\mathbf{X}^{-1} \in \mathbf{C}^{3 \times 3}$ and $\mathbf{n}\left(\frac{\beta}{\alpha}\Delta\right) \in \mathbf{C}^3$. \square

Remark 2.3.3. *In practice, the structure $\mathbf{n}\left(\frac{\beta}{\alpha}\Delta\right)$ is required in \mathbb{R}^3 and as a non-negative integer. For this reason, the modulus of the entries in $\mathbf{n}\left(\frac{\beta}{\alpha}\Delta\right)$ to the nearest integer should be used in real-life applications.*

3. Numerical Illustration

We illustrate the previous results in the Methodology with an example from a department in a university in Nigeria. We carryout all computations in the MATLAB environment. From the records [3], the sub-stochastic transition matrix is

$$\mathbf{P} = \begin{bmatrix} \frac{70}{83} & \frac{12}{83} & 0 \\ 0 & \frac{37}{46} & \frac{9}{46} \\ 0 & 0 & \frac{33}{34} \end{bmatrix}.$$

The time interval for GA's to obtain a higher degree from the university is 4 years. For this reason, we set the length of the interval, Δ , as 4 years. Let the grade-specific supervision completion rates for GA's within the time bound be $m_2 = 0.4$

and $m_3 = 1.2$. Suppose the system maintains a growth rate of 1% and adopts a recruitment policy of $p_{01} = 0.6$, $p_{02} = 0.4$ and $p_{03} = 0$. Then the transformation matrix \mathbf{A} is

$$\mathbf{A} = \begin{bmatrix} 0.8566 & 0.4060 & 1.2236 \\ 0.1534 & 0.8083 & 0.0158 \\ 0 & 0.1957 & 0.9706 \end{bmatrix}.$$

In order to evaluate $sp(\mathbf{A})$, we first compute the coefficients of the characteristic equation as $tr(\mathbf{A}) = 2.6355$, $S_2 = 2.2430$ and $det(\mathbf{A}) = 0.6457$. The trace $tr(\mathbf{A})$ lies within the interval defined by Proposition 2.2.2. Since $tr^2(\mathbf{A}) > 3S_2$, we determine λ from the formula

$$\lambda = \frac{1}{3}tr(\mathbf{A}) + r \cos\left(\frac{2k\pi}{3} + \frac{1}{3} \arccos \zeta\right)$$

in Proposition 2.2.1. Thus we obtain

$$sp(\mathbf{A}) = \{1.2687, 0.6834 - 0.2047z, 0.6834 + 0.2047z\}.$$

Since the eigenvalues of \mathbf{A} are simple, matrix \mathbf{A} is diagonalisable over the complex field \mathbf{C} . This is in line with Proposition 2.2.4. The matrix \mathbf{A} may be expressed as

$$\mathbf{A} = \mathbf{XDX}^{-1},$$

where

$$\mathbf{X} = \begin{bmatrix} 0.9260 & -0.8080 & -0.8080 \\ 0.3156 & 0.2523 + 0.4492z & 0.2523 - 0.4492z \\ 0.2071 & 0.0306 - 0.2842z & 0.0306 + 0.2842z \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} 1.2687 & 0 & 0 \\ 0 & 0.6834 + 0.2047z & 0 \\ 0 & 0 & 0.6834 - 0.2047z \end{bmatrix},$$

and

$$\mathbf{X}^{-1} = \begin{bmatrix} 0.3768 & 1.0124 & 1.6003 \\ -0.4029 - 0.0939z & 0.5801 - 0.4315z & 0.9170 + 1.0773z \\ -0.4029 + 0.0939z & 0.5801 + 0.4315z & 0.9170 - 1.0773z \end{bmatrix}.$$

The spectral radius of matrix \mathbf{A} , which is $\rho(\mathbf{A}) = 1.2687$, does not exceed the upper bound $(3 + |0.3105 \cos(0 + 0.7021z)|) = 3.3902$ as specified in Proposition 2.2.3. The value of $\rho(\mathbf{A})$ shows that $\lim_{\beta \rightarrow \infty} \mathbf{A}^\beta$ is infinitely large. Thus the grade structure $\mathbf{n}(t + \Delta)$ is unbounded as t becomes very large. Now, let the initial grade structure for the system be $\mathbf{n}(0) = [14, 6, 12]'$. The grade structure at $t = 0, \Delta$, are $\mathbf{n}(\Delta) = [29, 7, 13]'$ and $\mathbf{n}(2\Delta) = [44, 10, 14]'$. Now, suppose the structure of the system is required at a period $4\Delta/3$. Then $\mathbf{n}(4\Delta/3) = [34, 8, 13]'$. The realisation of the process satisfies the relation $N(\Delta) < N(4\Delta/3) < N(2\Delta)$ as the system is an expanding one.

4. Conclusion

This study provides a means of predicting a university manpower structure. The study provided a new representation for the transformation matrix of the system. A considerable amount of effort has been devoted to the properties of the transformation matrix. The transformation matrix is equivalent to the transition matrix in the manpower literature when the grade-specific supervision completion rates are equal to zero, that is $m_2 = m_3 = 0$. The introduction of the grade-specific supervision completion rates, m_2, m_3 , gives a better insight into the phenomenon and the output of supervisors on GA's. We opined that our model, which is a linear recurrence system, does succeed in unifying the stocks and flows of academic staff in a university system. The study is easy to follow and the theoretical sophistication is minimal. However, further work may be required to study the statistical properties of the linear recurrence system.

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About a Class of Analytic Functions Defined by Noor-Sălăgean Integral Operator

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ABSTRACT: In this paper we introduce a new integral operator as the convolution of the Noor and Sălăgean integral operators. With this integral operator we define the class $C_{NS}(\alpha)$, where $\alpha \in [0, 1)$ and we study some properties of this class.

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1. Introduction

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane \mathbb{C} and $H(U)$ denote the set of holomorphic (analytic) functions in U . We denote by

$$\mathcal{A} = \{f \in H(U) : f(0) = f'(0) - 1 = 0\}$$

and

$$S = \{f \in \mathcal{A} : f \text{ is univalent in } U\}.$$

We say that f is starlike in U if $f : U \rightarrow \mathbb{C}$ is univalent and $f(U)$ is a starlike domain in \mathbb{C} with respect to origin. It is well-known that $f \in \mathcal{A}$ is starlike in U if and only if

$$\operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, \quad \text{for all } z \in U.$$

The class of starlike functions with respect to origin is denoted by S^* .

Let T denote a subclass of \mathcal{A} consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

where $a_j \geq 0$, $j = 2, 3, \dots$ and $z \in U$. A function $f \in T$ is called a function with negative coefficients. For the class T , the followings are equivalent [6]:

- (i) $\sum_{j=2}^{\infty} ja_j \leq 1$,
- (ii) $f \in T \cap S$,
- (iii) $f \in T^*$, where $T^* = T \cap S^*$.

Let

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \quad a_j \geq 0, \quad j = 2, 3, \dots$$

and

$$g(z) = z - \sum_{j=2}^{\infty} b_j z^j, \quad b_j \geq 0, \quad j = 2, 3, \dots$$

then the convolution or the Hadamard product is defined by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j = (g * f)(z), \quad z \in U.$$

The study of operators plays an important role in geometric function theory. For $f \in H(U)$, $f(0) = 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, the I_S^n Sălăgean integral operator is defined as follows [7]:

- (i) $I_S^0 f(z) = f(z)$,
- (ii) $I_S^1 f(z) = I f(z) = \int_0^z f(t) t^{-1} dt$,
- (iii) $I_S^n f(z) = I_S(I_S^{n-1} f(z))$.

We remark that if f has the form (1.1), then

$$I_S^n f(z) = z - \sum_{j=2}^{\infty} \frac{a_j}{j^n} z^j, \quad (1.2)$$

where $n \in \mathbb{N}_0$.

In [5] Noor defined an integral operator $I_N^n : \mathcal{A} \rightarrow \mathcal{A}$ as follows

$$I_N^n f(z) = \frac{n+1}{z^n} \int_0^z t^{n-1} I_N^n(f(t)) dt, \quad (1.3)$$

where $n \in \mathbb{N}_0$.

Let $f_n(z) = \frac{z}{(1-z)^{n+1}}$ and let $f_n^{(-1)}(z)$ be defined such that

$$f_n^{(-1)}(z) * f_n(z) = \frac{z}{1-z}.$$

We note that

$$I_N^n f(z) = f_n^{(-1)}(z) * f(z) = \left[\frac{z}{(1-z)^{n+1}} \right]^{(-1)} * f(z).$$

We remark that if f has the form (1.1), then

$$I_N^n f(z) = z - \sum_{j=2}^{\infty} \frac{a_j}{C(n,j)} z^j, \tag{1.4}$$

where $C(n,j) = \frac{(n+j-1)!}{n!(j-1)!}$.

2. Preliminaries

The following definitions and lemmas will be required in the sequel.

Definition 2.1. [2, 3] Let f and g be analytic functions in U . We say that the function f is subordinate to the function g , if there exist a function w , which is analytic in U and for which $w(0) = 0$, $|w(z)| < 1$ for $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. We denote by \prec the subordination relation.

Definition 2.2. [3] Let Q be the class of analytic functions q in U which has the property that are analytic and injective on $\bar{U} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial U : \lim_{z \rightarrow \zeta} q(z) = \infty \right\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(q)$.

Lemma 2.1. [2, 3] Let $q \in Q$, with $q(0) = a$, and let $p(z) = a + a_n z^n + \dots$ be analytic in U with $p(z) \not\equiv a$ and $n \geq 1$. If $p \not\prec q$, then there are two points $z_0 = r_0 e^{i\theta_0} \in U$, and $\zeta_0 \in \partial U \setminus E(q)$ and a number $m \geq n \geq 1$ for which $p(U_{r_0}) \subset q(U)$,

- (i) $p(z_0) = q(\zeta_0)$
- (ii) $z_0 p'(z_0) = m \zeta_0 q'(\zeta_0)$
- (iii) $\operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq m \operatorname{Re} \left(\frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)} + 1 \right)$.

The following result is a particular case of Lemma 2.1.

Lemma 2.2. [2, 3] Let $p(z) = 1 + a_n z^n + \dots$ be analytic in U with $p(z) \not\equiv 1$ and $n \geq 1$. If $\operatorname{Re} p(z) \not\prec 0$, $z \in U$, then there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that

- (i) $p(z_0) = ix$
- (ii) $z_0 p'(z_0) = y \leq -\frac{n(x^2+1)}{2}$,
- (iii) $\operatorname{Re} z_0^2 p''(z_0) + z_0 p'(z_0) \leq 0$.

If $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, using the Noor and Sălăgean integral operators we define a new operator as follows:

$$I_{NS}^n f(z) = I_N^n f(z) * I_S^n f(z) = z - \sum_{j=2}^{\infty} \frac{a_j^2}{j^n C(n, j)} z^j, \quad (2.1)$$

where $C(n, j) = \frac{(n+j-1)!}{n!(j-1)!}$ and $n \in \mathbb{N}_0$.

Remark 2.1. Differentiate the relation (2.1), we get

$$[I_{NS}^n f(z)]' = 1 - \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n, j)} z^{j-1}. \quad (2.2)$$

Multiplicating the equality (2.2) with $\frac{z}{n}$ we obtain

$$\frac{z}{n} [I_{NS}^n f(z)]' = \frac{z}{n} - \sum_{j=2}^{\infty} \frac{a_j^2}{n j^{n-1} C(n, j)} z^j,$$

which is equivalent to

$$\frac{z}{n} [I_{NS}^n f(z)]' + \frac{z}{n} (n-1) = z - \sum_{j=2}^{\infty} \frac{a_j^2}{n j^{n-1} C(n, j)} z^j. \quad (2.3)$$

Now let $g \in T$ and $g(z) = z - \sum_{j=2}^{\infty} (n+j-1)z^j$. Then from (2.3), we obtain the following relation between $I_{NS}^{n-1} f(z)$ and $I_{NS}^n f(z)$ operators:

$$I_{NS}^{n-1} f(z) = \frac{z}{n} [I_{NS}^n f(z)]' * g(z) + \frac{n-1}{n} z * g(z). \quad (2.4)$$

Using the Noor-Sălăgean integral operator, we define the following class of analytic functions:

Definition 2.3. A function $f \in T$ belongs to the class $C_{NS}(\alpha)$ if

$$\operatorname{Re} \frac{z [I_{NS}^n f(z)]'}{I_{NS}^n f(z)} > \alpha, \quad (2.5)$$

where $\alpha \in [0, 1)$ and $z \in U$.

3. Main Results

Theorem 3.1. Let $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$. Then $f \in C_{NS}(\alpha)$ if and only if

$$\sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n, j)} \left[1 - \frac{\alpha}{j} \right] < 1 - \alpha. \quad (3.1)$$

Proof. Let $f \in C_{NS}(\alpha)$, then we have

$$\operatorname{Re} \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} > \alpha, \quad z \in U.$$

If $z \in [0, 1)$, we obtain

$$\frac{z - \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n, j)} z^j}{z - \sum_{j=2}^{\infty} \frac{a_j^2}{j^n C(n, j)} z^j} > \alpha. \quad (3.2)$$

Since the denominator of (3.2) is positive, the relation (3.2) is equivalent with

$$\alpha - 1 < \sum_{j=2}^{\infty} \left[\frac{\alpha a_j^2}{j^n C(n, j)} z^{j-1} - \frac{a_j^2}{j^{n-1} C(n, j)} z^{j-1} \right],$$

and finally we get

$$\alpha - 1 < \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n, j)} z^{j-1} \left[\frac{\alpha}{j} - 1 \right].$$

Considering $z \rightarrow 1^-$ along to the real axis, we get:

$$\alpha - 1 < \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n, j)} \left[\frac{\alpha}{j} - 1 \right].$$

To prove the reciproc implication we consider f with the form (1.1) and for which the (3.1) inequality holds.

The condition $\operatorname{Re} \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} > \alpha$ is equivalent to

$$\alpha - \operatorname{Re} \left(\frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right) < 1.$$

We have

$$\begin{aligned} \alpha - \operatorname{Re} \left(\frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right) &\leq \alpha + \left| \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right| \\ &= \alpha + \left| \frac{\sum_{j=2}^{\infty} \frac{a_j^2}{j^n C(n, j)} z^j - \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n, j)} z^j}{z - \sum_{j=2}^{\infty} \frac{a_j^2}{j^n C(n, j)} z^j} \right| = \alpha + \left| \frac{\sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n, j)} z^{j-1} \left[\frac{1}{j} - 1 \right]}{1 - \sum_{j=2}^{\infty} \frac{a_j^2}{j^n C(n, j)} z^{j-1}} \right| \\ &\leq \alpha + \frac{\sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n, j)} |z|^{j-1} \left| \frac{1}{j} - 1 \right|}{1 - \sum_{j=2}^{\infty} \frac{a_j^2}{j^n C(n, j)} |z|^{j-1}} < \alpha + \frac{\sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1} C(n, j)} \left[1 - \frac{1}{j} \right]}{1 - \sum_{j=2}^{\infty} \frac{a_j^2}{j^n C(n, j)}} \end{aligned}$$

$$\begin{aligned} & \alpha + \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left[1 - \frac{1}{j} - \frac{\alpha}{j}\right] \\ &= \frac{\alpha + \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left[1 - \frac{1}{j} - \frac{\alpha}{j}\right]}{1 - \sum_{j=2}^{\infty} \frac{a_j^2}{j^n C(n,j)}}. \end{aligned}$$

To finish our proof, we need to show

$$\frac{\alpha + \sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left[1 - \frac{1}{j} - \frac{\alpha}{j}\right]}{1 - \sum_{j=2}^{\infty} \frac{a_j^2}{j^n C(n,j)}} < 1. \quad (3.3)$$

The (3.3) inequality is equivalent to

$$\sum_{j=2}^{\infty} \frac{a_j^2}{j^{n-1}C(n,j)} \left[1 - \frac{\alpha}{j}\right] < 1 - \alpha, \quad (3.4)$$

which is the (3.1) condition. \square

Let $E_{NS}(\alpha)$ be a subclass of $C_{NS}(\alpha)$. The class is defined as follows:

$$E_{NS}(\alpha) = \left\{ f \in T : \left| \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right| < 1 - 2\alpha \text{ and } \alpha \in \left(0, \frac{1}{2}\right) \right\}. \quad (3.5)$$

Theorem 3.2. *Let $f \in T$ of the form (1.1). If $f \in E_{NS}(\alpha)$, then $\operatorname{Re} \frac{I_{NS}^n f(z)}{z} > 0$.*

Proof. Suppose $f \in E_{NS}(\alpha)$. Then

$$\left| \frac{z[I_{NS}^n f(z)]'}{I_{NS}^n f(z)} - 1 \right| < 1 - 2\alpha. \quad (3.6)$$

Let

$$I_{NS}^n f(z) = zp(z). \quad (3.7)$$

Differentiate (3.7), we obtain

$$[I_{NS}^n f(z)]' = zp'(z) + p(z). \quad (3.8)$$

Then (3.6) is equivalent to

$$\left| \frac{zp'(z)}{p(z)} \right| < 1 - 2\alpha.$$

If the condition $\operatorname{Re} p(z) = \operatorname{Re} \frac{I_{NS}^n f(z)}{z} > 0$ does not hold, then according to Lemma 2.2, there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that

$$p(z_0) = ix$$

and

$$z_0 p'(z_0) = y \leq -\frac{1+x^2}{2}.$$

These inequalities imply

$$\left| \frac{z_0 p'(z_0)}{p(z_0)} \right| = \left| \frac{y}{ix} \right| \geq \left| \frac{\frac{1}{2}(1+x^2)}{x} \right| = \left| \frac{1}{2} \left(x + \frac{1}{x} \right) \right| \geq 1 - 2\alpha.$$

The above inequality contradicts (3.6) and consequently

$$\operatorname{Re} p(z) = \operatorname{Re} \frac{I_{NS}^n f(z)}{z} > 0,$$

where $z \in U$. □

Theorem 3.3. *Let*

$$F(z) = I_c f(z) = \frac{c+1}{z^c} \int_0^z f(t) t^{c-1} dt, c \in \mathbb{N}.$$

If $f \in C_{NS}(\alpha)$, then $F = I_c(f) \in C_{NS}(\beta)$, where

$$\beta = \beta(\alpha, 2) = 1 - \frac{(1-\alpha)(c+1)^2}{(c+2)^2(2-\alpha) - (c+1)^2(1-\alpha)} \tag{3.9}$$

and $\beta > \alpha$, $\alpha \in [0, 1)$.

Proof. Suppose $f \in C_{NS}(\alpha)$. Then by Theorem 3.1 we have

$$\sum_{j=2}^{\infty} \frac{a_j^2(j-\alpha)}{j^n C(n, j)(1-\alpha)} < 1.$$

We know that if f has the form (1.1), then

$$F(z) = \frac{c+1}{z^c} \int_0^z f(t) t^{c-1} dt = z - \sum_{j=2}^{\infty} \frac{c+1}{c+j} a_j z^j,$$

and to prove that $F \in C_{NS}(\beta)$ is sufficient to have

$$\sum_{j=2}^{\infty} \frac{j-\beta}{j^n C(n, j)(1-\beta)} \left(\frac{c+1}{c+j} \right)^2 a_j^2 < 1.$$

This last inequality is implied by

$$\frac{j-\beta}{1-\beta} \cdot \frac{(c+1)^2 a_j^2}{j^n C(n, j)(c+j)^2} \leq \frac{j-\alpha}{1-\alpha} \cdot \frac{a_j^2}{j^n C(n, j)}, \tag{3.10}$$

for all $j \in \mathbb{N}$ and $j \geq 2$.

From (3.10) we deduce that

$$\beta \leq 1 - \frac{(1-\alpha)(c+1)^2(j-1)}{(c+j)^2(j-\alpha) - (c+1)^2(1-\alpha)} = \beta(\alpha, j),$$

$j \in \mathbb{N}$, $j \geq 2$. We will prove that

$$\beta(\alpha, j) \geq \beta(\alpha, 2), \quad j \in \mathbb{N}, \quad j \geq 2.$$

Let consider the function $\varphi : [2, \infty) \rightarrow \mathbb{R}$,

$$\varphi(x) = \frac{x-1}{(x+c)^2(x-\alpha) - (c+1)^2(1-\alpha)}, \quad x \in [2, \infty).$$

Then

$$\varphi'(x) = \frac{g(x)}{[(x+c)^2(x-\alpha) - (c+1)^2(1-\alpha)]^2},$$

where $g(x) = -2x^3 + (3-2c-\alpha)x^2 + (4c-2\alpha)x - 2c - (1-\alpha)$.

We have

$$\begin{aligned} g'(x) &= -6x^2 + 2(3-2c-\alpha)x + 4c - 2\alpha, \\ g''(x) &= -12x + 6 - 4c - 2\alpha < 0, \end{aligned}$$

$x \in [2, \infty)$. Then

$$g'(x) \leq g'(2) = -12 - 4c - 6\alpha < 0, \quad x \in [2, \infty)$$

and

$$g(x) \leq g(2) = -4 - 8\alpha - 2c - (1-\alpha) < 0, \quad x \in [2, \infty).$$

We obtain $\varphi'(x) < 0$, $x \in [2, \infty)$ and from this

$$\beta(\alpha, j) = 1 - \varphi(j)(1-\alpha)(c+1)^2 \geq 1 - \varphi(2)(1-\alpha)(c+1)^2 = \beta(\alpha, 2)$$

where $\beta(\alpha, 2)$ is given by (3.9). Finally $\beta > \alpha$ is equivalent to

$$1 - \alpha > \frac{(1-\alpha)(c+1)^2}{(c+2)^2(2-\alpha) - (c+1)^2(1-\alpha)}.$$

□

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A Refinement of Schwarz's Lemma and its Applications

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ABSTRACT: By using the value of the second derivative of the function at 0, along with the values of the function and its first derivative at 0, we have obtained a refinement of well known Schwarz's lemma and have used this refinement to obtain refinements, of Aziz and Rather's inequalities [2004] for a polynomial of degree n having no zeros in $|z| < k$, ($k \geq 1$).

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1. Introduction and statement of results

Concerning the values of a function, analytic in the interior of a disc, we have the following well known result, known as Schwarz's lemma [2, p. 189-190].

Theorem A. *If $f(z)$ is analytic in $|z| < 1$, where it satisfies the inequality $|f(z)| \leq 1$, and if $f(0) = 0$, then the inequality*

$$|f(z)| \leq |z|$$

holds whenever $|z| < 1$. Moreover equality can occur only when $f(z) = ze^{i\alpha}$, where α is a real constant.

There is a generalization of Theorem A, known as generalization of Schwarz's lemma [6, p. 212], which can be stated as

Theorem B. *If $f(z)$ is analytic and $|f(z)| \leq 1$, in $|z| < 1$ then*

$$|f(z)| \leq \frac{|z| + |a|}{|a||z| + 1}, \quad |z| < 1,$$

where

$$a = f(0).$$

Govil et al. [4] obtained the following refinement of Theorem B, by using the value of the first derivative of the function at 0, along with the value of the function at 0.

Theorem C. *If $f(z)$ is analytic and $|f(z)| \leq 1$, in $|z| < 1$ then*

$$|f(z)| \leq \begin{cases} \frac{(1-|a|)|z|^2+|bz|+|a|(1-|a|)}{|a|(1-|a|)|z|^2+|bz|+(1-|a|)}, & |a| < 1, \\ 1 & , |a| = 1, \end{cases}, \quad |z| < 1, \quad (1.1)$$

where

$$a = f(0), \quad b = f'(0).$$

The example

$$f(z) = (a + \frac{b}{1+a}z - z^2)/(1 - \frac{b}{1+a}z - az^2)$$

shows that the estimate is sharp.

In this paper we have firstly obtained a refinement of Theorem C, thereby giving a refinement of Theorem A also, by using the value of the second derivative of the function at 0, along with the values of the function and its first derivative at 0. More precisely we have proved

Theorem 1. *Let $f(z)$ be analytic in $|z| < 1$, with*

$$|f(z)| \leq 1, \quad |z| < 1,$$

$$a = f(0), \quad b = f'(0), \quad c = f''(0), \quad (1.2)$$

$$\gamma = \begin{cases} \arg \bar{a}, & a \neq 0, \\ \text{any value}, & a = 0, \end{cases}, \quad (1.3)$$

$$A = 2(1 - |a|^2)(1 - |a|^2 - |b|), \quad (1.4)$$

$$B = 2|a||b|(1 - |a|^2 - |b|) + ||a|^2c - 2e^{i\gamma}|a|b^2 - c|, \quad (1.5)$$

$$C = 2|b|(1 - |a|^2 - |b|) + |a||a|^2c - 2e^{i\gamma}|a|b^2 - c| \quad (1.6)$$

and

$$D = 2|a|(1 - |a|^2)(1 - |a|^2 - |b|). \quad (1.7)$$

Then

$$|f(z)| \leq \begin{cases} \frac{A|z|^3+B|z|^2+C|z|+D}{D|z|^3+C|z|^2+B|z|+A}, & |a| < 1 \text{ and } |b| < 1 - |a|^2, \\ \frac{|z|+|a|}{|a||z|+1} & , |a| < 1 \text{ and } |b| = 1 - |a|^2, \\ 1 & , |a| = 1, \end{cases}, \quad |z| < 1. \quad (1.8)$$

Remark 1. By using the result [5, p. 172, exercise # 9] one can show that Theorem 1 is a refinement of Theorem C.

Further for a polynomial $p(z)$, let

$$\begin{aligned}\|p\|_q &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^q d\theta \right\}^{1/q}, \quad q > 0, \\ \|p\|_\infty &= \max_{|z|=1} |p(z)|.\end{aligned}$$

Then secondly we have used Theorem 1 to obtain a refinement of Aziz and Rather's result [1, Theorem 1 and Corollary 1]

Theorem D. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n , having no zeros in $|z| < k$, ($k \geq 1$), with

$$1 \leq s < n$$

and

$$\delta_{k,s} = \frac{C(n,s)|a_0|k^{s+1} + |a_s|k^{2s}}{C(n,s)|a_0| + |a_s|k^{s+1}}. \quad (1.9)$$

Then

$$\|P^{(s)}\|_q \leq \frac{n(n-1)\dots(n-s+1)}{\|\delta_{k,s} + z\|_q} \|P\|_q, \quad q > 0$$

and

$$\|P^{(s)}\|_\infty \leq n(n-1)\dots(n-s+1) \left\{ \frac{C(n,s)|a_0| + |a_s|k^{s+1}}{C(n,s)|a_0|(1+k^{s+1}) + k^{s+1}(k^{s-1}+1)|a_s|} \right\} \|P\|_\infty.$$

More precisely we have proved

Theorem 2. Let $P(z) = \sum_{j=0}^n a_j z^j$ be a polynomial of degree n , having no zeros in $|z| < k$, ($k \geq 1$), with

$$1 \leq s < n,$$

$$E = \frac{C(n,s)|a_0| + |a_s|k^{s+1}}{C(n,s)|a_0|k + |a_s|k^s}, \quad (1.10)$$

$$\delta_{k,s} = k^s/E, \quad (\text{by (1.9)}),$$

$$F = |n(s+1)a_0a_{s+1} - (n-s)a_1a_s|, \quad (1.11)$$

$$\begin{aligned}H &= n|a_0|^2 \{C(n,s)\}^2 + |a_0||a_s|nC(n,s)k^s(k^2-1) - n|a_s|^2k^{2s+2} \\ &\quad + k^{s+2}C(n,s)F, \\ J &= nk^2 \{C(n,s)\}^2 |a_0|^2 - nC(n,s)|a_0||a_s|k^s(k^2-1) - n|a_s|^2k^{2s} \\ &\quad + k^{s+2}C(n,s)F,\end{aligned}$$

$$d_{k,s} = \begin{cases} k^s J/H, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| < 1, \\ k^s, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| = 1, \end{cases}$$

$$a_{n+1} = 0,$$

$$\gamma = \begin{cases} \arg \left(\frac{k^s \overline{a_s}}{C(n,s)a_0} \right), \frac{k^s a_s}{C(n,s)a_0} \neq 0, \\ \text{any value}, \frac{k^s a_s}{C(n,s)a_0} = 0, \end{cases}$$

$$\begin{aligned} G &= |a_0 \{ n^2(n-1)a_0^2(s+2)(s+1)a_{s+2} - 2n(n-s)a_0((n-s-1)a_2a_s + \\ &\quad (n-1)(s+1)a_1a_{s+1}) + 2(n-1)(n-s)^2a_1^2a_s \} \{ k^{2s}|a_s|^2 - \\ &\quad (C(n,s))^2|a_0|^2 \} - 2e^{i\gamma}(n-1)|a_0||a_s|k^{2s}\{ n(s+1)a_0a_{s+1} - \\ &\quad (n-s)a_1a_s \}^2|, \end{aligned} \quad (1.12)$$

$$\begin{aligned} L &= 2n^2(n-1)\{ (C(n,s))^5|a_0|^7 + (C(n,s))^4k^{s+3}|a_0|^6|a_s| - \\ &\quad 2(C(n,s))^3k^{2s}|a_0|^5|a_s|^2 - 2(C(n,s))^2k^{3s+3}|a_0|^4|a_s|^3 \\ &\quad + C(n,s)k^{4s}|a_0|^3|a_s|^4 + k^{5s+3}|a_0|^2|a_s|^5 \} + \\ &\quad 2n(n-1)C(n,s)k^{s+1}|a_0|^2F(k^2-1)\{ |a_0|^3(C(n,s))^3 - \\ &\quad (C(n,s))^2|a_0|^2|a_s|k^{s+1} - (C(n,s))|a_0||a_s|^2k^{2s} + \\ &\quad |a_s|^3k^{3s+1} \} + C(n,s)k^{s+3}G\{ C(n,s)|a_0| + |a_s|k^{s+1} \} - \\ &\quad 2(n-1)(C(n,s))^2k^{2s+3}F^2|a_0|^2\{ kC(n,s)|a_0| + k^s|a_s| \}, \end{aligned} \quad (1.13)$$

$$\begin{aligned} S &= 2n^2(n-1)|a_0|^2\{ (C(n,s))^5|a_0|^5k^3 + (C(n,s))^4|a_0|^4|a_s|k^s - \\ &\quad 2(C(n,s))^3|a_0|^3|a_s|^2k^{2s+3} - 2(C(n,s))^2|a_0|^2|a_s|^3k^{3s} + \\ &\quad C(n,s)|a_0||a_s|^4k^{4s+3} + |a_s|^5k^{5s} \} + 2n(n-1)|a_0|^2k^{s+1}C(n,s)(k^2-1) \\ &\quad F\{ -(C(n,s))^3k|a_0|^3 + (C(n,s))^2k^s|a_0|^2|a_s| + C(n,s)k^{2s+1}|a_0||a_s|^2 \\ &\quad - |a_s|^3k^{3s} \} - 2(n-1)(C(n,s))^2k^{2s+3}|a_0|^2F^2\{ |a_0|C(n,s) + \\ &\quad k^{s+1}|a_s| \} + Gk^{s+3}C(n,s)\{ |a_0|k + |a_s|k^s \}, \end{aligned} \quad (1.14)$$

$$\gamma_{k,s} = \begin{cases} \frac{k^s S}{L}, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1}F}{n|a_0|^2C(n,s)} < 1 - \frac{k^{2s}}{(C(n,s))^2} \left| \frac{a_s}{a_0} \right|^2, & (1.15) \\ \frac{k^s}{E}, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1}F}{n|a_0|^2C(n,s)} = 1 - \frac{k^{2s}}{(C(n,s))^2} \left| \frac{a_s}{a_0} \right|^2, & (1.16) \\ k^s, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| = 1. & (1.17) \end{cases}$$

Then for $q > 0$

$$\begin{aligned}
 \|P^{(s)}\|_q &\leq \frac{n(n-1)\dots(n-s+1)}{\|\gamma_{k,s} + z\|_q} \|P\|_q, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{\|d_{k,s} + z\|_q} \|P\|_q, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{\|\delta_{k,s} + z\|_q} \|P\|_q, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{\|k^s + z\|_q} \|P\|_q.
 \end{aligned} \tag{1.18}$$

Remark 2. With the additional assumption

$$a_{n+2} = 0,$$

one can show that Theorem 2 is true for $s = n$ also.

By letting $q \rightarrow \infty$ in (1.18), we obtain

Corollary 1. Under the same hypotheses as in Theorem 2

$$\begin{aligned}
 \|P^{(s)}\|_\infty &\leq \frac{n(n-1)\dots(n-s+1)}{(\gamma_{k,s} + 1)} \|P\|_\infty, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{(d_{k,s} + 1)} \|P\|_\infty, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{(\delta_{k,s} + 1)} \|P\|_\infty, \\
 &\leq \frac{n(n-1)\dots(n-s+1)}{(k^s + 1)} \|P\|_\infty.
 \end{aligned}$$

Remark 3. With the additional assumption

$$a_{n+2} = 0,$$

one can show that Corollary 1 is true for $s = n$ also.

Remark 4. Corollary 1 is also a refinement of Govil and Rahman's result [3, Theorem 4].

2. Lemmas

For the proofs of the theorems we require the following lemmas.

Lemma 1. *If $f(z)$ is analytic and $|f(z)| \leq 1$, in $|z| \leq 1$ then*

$$|f(z)| \leq \frac{|z| + |a|}{1 + |a||z|}, \quad |z| \leq 1,$$

where

$$a = f(0).$$

Proof of Lemma 1. It easily follows from Theorem B.

Lemma 2. *If $f(z)$ is analytic and $|f(z)| \leq 1$, in $|z| \leq 1$ then*

$$|f(z)| \leq \left\{ \begin{array}{l} \frac{(1-|a|)|z|^2+|bz|+|a|(1-|a|)}{|a|(1-|a|)|z|^2+|bz|+(1-|a|)}, |a| < 1, \\ 1, |a| = 1, \end{array} \right\}, \quad |z| \leq 1,$$

where

$$a = f(0), \quad b = f'(0).$$

Proof of Lemma 2. It easily follows from Theorem C.

Remark 5. By using the result [5, p.172, exercise # 9] one can show that Lemma 2 is a refinement of Lemma 1.

Lemma 3. *Let $f(z)$ be analytic in $|z| \leq 1$, with*

$$|f(z)| \leq 1, \quad |z| \leq 1,$$

$$a = f(0), \quad b = f'(0), \quad c = f''(0),$$

$$\gamma = \left\{ \begin{array}{l} \arg \bar{a}, \quad a \neq 0, \\ \text{any value}, \quad a = 0, \end{array} \right\}.$$

Then

$$|f(z)| \leq \left\{ \begin{array}{l} \frac{A|z|^3+B|z|^2+C|z|+D}{D|z|^3+C|z|^2+B|z|+A}, |a| < 1 \text{ and } |b| < 1 - |a|^2, \\ \frac{|z|+|a|}{1+|a||z|}, |a| < 1 \text{ and } |b| = 1 - |a|^2, \\ 1, |a| = 1, \end{array} \right\}, \quad |z| \leq 1,$$

where A, B, C and D are, as in Theorem 1.

Proof of Lemma 3. It easily follows from Theorem 1.

Remark 6. By Remark 1 one can say that Lemma 3 is a refinement of Lemma 2.

Lemma 4. *Under the same hypotheses as in Theorem 2*

$$k^s |P^{(s)}(z)| \leq \delta_{k,s} |P^{(s)}(z)| \leq d_{k,s} |P^{(s)}(z)| \leq \gamma_{k,s} |P^{(s)}(z)| \leq |Q^{(s)}(z)|, \quad |z| = 1, \quad (2.1)$$

where

$$Q(z) = z^n \overline{P(1/\bar{z})}. \quad (2.2)$$

Proof of Lemma 4. Let

$$\begin{aligned} G(z) &= z^n \overline{P(k/\bar{z})} = k^n Q(z/k), \\ H(z) &= z^{n-s} \overline{G^{(s)}(1/\bar{z})} = (kz)^{n-s} \overline{Q^{(s)}(1/(k\bar{z}))}. \end{aligned} \quad (2.3)$$

Then by using the result [1, inequality 32] we get

$$k^s |P^{(s)}(kz)| \leq |H(z)|, \quad |z| = 1, \quad (2.4)$$

with $H(z)$ having all its zeros in $|z| \geq 1$. Further let

$$H(z) = \Phi(z)H_1(z), \quad (2.5)$$

with

$$H_1(z) \neq 0, \quad |z| = 1$$

and

$$\Phi(z) = \begin{cases} 1 & ; H(z) \neq 0 \text{ on } |z| = 1, \\ \prod_{\gamma=1}^m (z - z_\gamma), |z_\gamma| = 1 \forall \gamma; & H(z) \text{ has certain zeros on } |z| = 1. \end{cases} \quad (2.6)$$

Then by (2.4), (2.5) and (2.6) we have

$$k^s P^{(s)}(kz) = \Phi(z)P_1(z), \quad (2.7)$$

with

$$|P_1(z)| \leq |H_1(z)|, \quad |z| = 1. \quad (2.8)$$

Now

$$f(z) = \frac{P_1(z)}{H_1(z)} \quad (2.9)$$

is analytic in $|z| \leq 1$, with

$$|f(z)| \leq 1, \quad |z| = 1, \text{ (by (2.8))}$$

and therefore

$$|f(z)| \leq 1, \quad |z| \leq 1, \text{ (by maximum modulus principle),} \quad (2.10)$$

with

$$|f(z)| \leq \frac{|z| + |a|}{1 + |a||z|}, \quad |z| \leq 1, \text{ (a refinement of (2.10)),} \quad (2.11)$$

$$|f(z)| \leq \left\{ \begin{array}{l} \frac{(1-|a|)|z|^2 + |b||z| + |a|(1-|a|)}{|a|(1-|a|)|z|^2 + |b||z| + (1-|a|)}, |a| < 1, \\ 1, |a| = 1, \end{array} \right\}, \quad |z| \leq 1,$$

$$\text{(a refinement of (2.11)), (by Remark 5)} \quad (2.12)$$

and

$$|f(z)| \leq \left\{ \begin{array}{l} \frac{A|z|^3 + B|z|^2 + C|z| + D}{D|z|^3 + C|z|^2 + B|z| + A}, |a| < 1 \text{ and } |b| < 1 - |a|^2, \\ \frac{|z| + |a|}{1 + |a||z|}, |a| < 1 \text{ and } |b| = 1 - |a|^2, \\ 1, |a| = 1, \end{array} \right\}, \quad |z| \leq 1,$$

$$\text{(a refinement of (2.12)), (by Remark 6).} \quad (2.13)$$

Further if

$$g(z) = \frac{k^s P^{(s)}(kz)}{H(z)}$$

then

$$\begin{aligned} a &= f(0), \text{ (by (1.2))}, \\ &= g(0), \text{ (by (2.5), (2.7) and (2.9))}, \\ &= \frac{k^s a_s}{C(n, s) a_0}, \end{aligned} \quad (2.14)$$

$$\begin{aligned} b &= f'(0), \text{ (by (1.2))}, \\ &= g'(0), \text{ (by (2.5), (2.7) and (2.9))}, \\ &= \frac{k^{s+1}}{n a_0^2 C(n, s)} \{n(s+1) a_0 a_{s+1} - (n-s) a_1 a_s\}, \end{aligned} \quad (2.15)$$

$$\begin{aligned} c &= f''(0), \text{ (by (1.2))}, \\ &= g''(0), \text{ (by (2.5), (2.7) and (2.9))}, \\ &= \frac{k^{s+2}}{n^2 (n-1) a_0^3 C(n, s)} \{n^2 (n-1) a_0^2 (s+1) (s+2) a_{s+2} - \\ &\quad 2n(n-s) a_0 ((n-s-1) a_2 a_s + \\ &\quad (n-1)(s+1) a_1 a_{s+1}) + 2(n-1)(n-s)^2 a_1^2 a_s\} \end{aligned} \quad (2.16)$$

and on using (2.13), with

$$z = \frac{1}{k} e^{i\theta}, \quad 0 \leq \theta \leq 2\pi,$$

we get

$$\left| f\left(\frac{1}{k} e^{i\theta}\right) \right| \leq \left\{ \begin{array}{l} L/S, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1} F}{n |a_0|^2 C(n, s)} < 1 - \frac{k^{2s}}{(C(n, s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ E, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1} F}{n |a_0|^2 C(n, s)} = 1 - \frac{k^{2s}}{(C(n, s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ 1, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| = 1, \end{array} \right\},$$

(by (2.14), (2.15), (2.16), (1.3), (1.4), (1.5), (1.6), (1.7), (1.11), (1.12), (1.13), (1.14) and (1.10)),

i.e.

$$\left| P_1\left(\frac{1}{k} e^{i\theta}\right) \right| \leq \left\{ \begin{array}{l} L/S, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1} F}{n |a_0|^2 C(n, s)} < 1 - \frac{k^{2s}}{(C(n, s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ E, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1} F}{n |a_0|^2 C(n, s)} = 1 - \frac{k^{2s}}{(C(n, s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ 1, \frac{k^s}{C(n, s)} \left| \frac{a_s}{a_0} \right| = 1, \end{array} \right\} \left| H_1\left(\frac{1}{k} e^{i\theta}\right) \right|, \quad (\text{by (2.9)}),$$

which, by (2.5) and (2.7), implies that

$$k^s |P^{(s)}(e^{i\theta})| \leq \left\{ \begin{array}{l} L/S, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1}F}{n|a_0|^2 C(n,s)} < 1 - \frac{k^{2s}}{(C(n,s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ E, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| < 1 \text{ and } \frac{k^{s+1}F}{n|a_0|^2 C(n,s)} = 1 - \frac{k^{2s}}{(C(n,s))^2} \left| \frac{a_s}{a_0} \right|^2, \\ 1, \frac{k^s}{C(n,s)} \left| \frac{a_s}{a_0} \right| = 1, \end{array} \right\} \left| H\left(\frac{1}{k}e^{i\theta}\right) \right|. \quad (2.17)$$

And as

$$H(z/k) = z^{n-s} \overline{Q^{(s)}(1/\bar{z})}, \quad (\text{by (2.3)}),$$

we get, by using (1.15), (1.16) and (1.17) in (2.17), that

$$\gamma_{k,s} |P^{(s)}(e^{i\theta})| \leq |Q^{(s)}(e^{i\theta})|, \quad 0 \leq \theta \leq 2\pi. \quad (2.18)$$

Now as we have obtained (2.18) by using (2.13), we can similarly obtain

$$d_{k,s} |P^{(s)}(e^{i\theta})| \leq |Q^{(s)}(e^{i\theta})|, \quad 0 \leq \theta \leq 2\pi,$$

$$\delta_{k,s} |P^{(s)}(e^{i\theta})| \leq |Q^{(s)}(e^{i\theta})|, \quad 0 \leq \theta \leq 2\pi$$

and

$$k^s |P^{(s)}(e^{i\theta})| \leq |Q^{(s)}(e^{i\theta})|, \quad 0 \leq \theta \leq 2\pi,$$

by using (2.12), (2.11) and (2.10) respectively. Further

$$k^s \leq \delta_{k,s} \leq d_{k,s} \leq \gamma_{k,s} \quad (2.19)$$

follows from the fact

$$\left\{ \begin{array}{l} \frac{A|z|^3+B|z|^2+C|z|+D}{D|z|^3+C|z|^2+B|z|+A}, |a| < 1 \text{ and } |b| < 1 - |a|^2, \\ \frac{|z|+|a|}{1+|a||z|}, |a| < 1 \text{ and } |b| = 1 - |a|^2, \\ 1, |a| = 1, \end{array} \right\} \leq \left\{ \begin{array}{l} \frac{(1-|a|)|z|^2+|b||z|+|a|(1-|a|)}{|a|(1-|a|)|z|^2+|b||z|+(1-|a|)}, |a| < 1, \\ 1, |a| = 1, \end{array} \right\} \leq \frac{|z|+|a|}{1+|a||z|} \leq 1, \quad |z| \leq 1,$$

(by (2.13), (2.12), (2.11) and (2.10))

and the way, L. H. S. of inequality (2.17) was obtained from R. H. S. of inequality (2.18). This completes the proof of Lemma 4. \square

Remark 7. With the additional assumption

$$a_{n+2} = 0,$$

one can show that Lemma 4 is true for $s = n$ also.

3. Proofs of the theorems

Proof of Theorem 1. If

$$|f(0)| = 1$$

then result follows trivially. Therefore from now onwards we will assume that

$$|f(0)| < 1.$$

Now we consider the function

$$g(z) = \frac{e^{i\gamma}f(z) - |f(0)|}{|f(0)|e^{i\gamma}f(z) - 1}, \quad (3.1)$$

which is analytic in $|z| < 1$, with

$$\begin{aligned} |g(z)| &\leq 1, \quad |z| < 1, \\ g(0) &= 0, \\ |g(z)| &\leq |z|, \quad |z| < 1, \text{ (by Schwarz's lemma),} \\ g'(0) &= \frac{e^{i\gamma}f'(0)}{|f(0)|^2 - 1}, \end{aligned} \quad (3.2)$$

$$g''(0) = \frac{e^{i\gamma}}{(|f(0)|^2 - 1)^2} [f''(0)\{|f(0)|^2 - 1\} - 2e^{i\gamma}|f(0)|(f'(0))^2] \quad (3.3)$$

and the function

$$\Phi(z) = \begin{cases} \frac{g(z)}{z}, & 0 < |z| < 1, \\ g'(0), & z = 0, \end{cases} \quad (3.4)$$

$$(3.5)$$

which is analytic in $|z| < 1$, with

$$\begin{aligned} |\Phi(z)| &\leq 1, \quad |z| < 1, \\ \Phi'(0) &= \frac{1}{2}g''(0). \end{aligned} \quad (3.6)$$

We apply Theorem C to $\Phi(z)$. If

$$|\Phi(0)| = 1$$

then by (1.1) and (3.4)

$$\left| \frac{g(z)}{z} \right| \leq 1, \quad 0 < |z| < 1,$$

i.e.

$$\left| \frac{e^{i\gamma}f(z) - |f(0)|}{|f(0)|e^{i\gamma}f(z) - 1} \right| \leq |z|, \quad |z| < 1, \text{ (by (3.1)),}$$

which implies

$$|e^{i\gamma} f(z)| \leq \frac{|z| + |f(0)|}{1 + |f(0)||z|}, \quad |z| < 1,$$

thereby proving second part of (1.8). And if

$$|\Phi(0)| < 1$$

then by (1.1) and (3.4)

$$\left| \frac{g(z)}{z} \right| \leq \frac{(1 - |\Phi(0)|)|z|^2 + |\Phi'(0)||z| + |\Phi(0)|(1 - |\Phi(0)|)}{|\Phi(0)|(1 - |\Phi(0)|)|z|^2 + |\Phi'(0)||z| + (1 - |\Phi(0)|)}, \quad 0 < |z| < 1,$$

i.e.

$$\begin{aligned} \left| \frac{e^{i\gamma} f(z) - |f(0)|}{|f(0)|e^{i\gamma} f(z) - 1} \right| &\leq |z| \frac{(1 - |\Phi(0)|)|z|^2 + |\Phi'(0)||z| + |\Phi(0)|(1 - |\Phi(0)|)}{|\Phi(0)|(1 - |\Phi(0)|)|z|^2 + |\Phi'(0)||z| + (1 - |\Phi(0)|)}, \quad |z| < 1, \\ &= E_0, \text{ (say)}, \quad |z| < 1, \end{aligned} \quad \text{(by (3.1)),}$$

which implies

$$\begin{aligned} |e^{i\gamma} f(z)| &\leq \frac{E_0 + |f(0)|}{1 + E_0|f(0)|}, \quad |z| < 1, \\ &= \frac{A|z|^3 + B|z|^2 + C|z| + D}{D|z|^3 + C|z|^2 + B|z| + A}, \quad |z| < 1, \\ &\text{(by (3.5), (3.6), (3.2), (3.3), (1.2), (1.4), (1.5), (1.6) and (1.7)),} \end{aligned}$$

thereby proving first part of (1.8). This completes the proof of Theorem 1. \square

Proof of Theorem 2. As Aziz and Rather [1, Theorem 1] had obtained

$$\|P^{(s)}\|_q \leq \frac{n(n-1)\dots(n-s+1)}{\|\delta_{k,s} + z\|_q} \|P\|_q,$$

by using

$$\delta_{k,s}|P^{(s)}(z)| \leq |Q^{(s)}(z)|, \quad |z| = 1, \text{ (by (2.2) and (2.1)),}$$

one can similarly obtain

$$\begin{aligned} \|P^{(s)}\|_q &\leq \frac{n(n-1)\dots(n-s+1)}{\|\gamma_{k,s} + z\|_q} \|P\|_q, \\ \|P^{(s)}\|_q &\leq \frac{n(n-1)\dots(n-s+1)}{\|d_{k,s} + z\|_q} \|P\|_q \end{aligned}$$

and

$$\|P^{(s)}\|_q \leq \frac{n(n-1)\dots(n-s+1)}{\|k^s + z\|_q} \|P\|_q,$$

by using

$$\gamma_{k,s}|P^{(s)}(z)| \leq |Q^{(s)}(z)|, \quad |z| = 1, \quad (\text{by (2.2) and (2.1)}),$$

$$d_{k,s}|P^{(s)}(z)| \leq |Q^{(s)}(z)|, \quad |z| = 1, \quad (\text{by (2.2) and (2.1)})$$

and

$$k^s|P^{(s)}(z)| \leq |Q^{(s)}(z)|, \quad |z| = 1, \quad (\text{by (2.2) and (2.1)})$$

respectively. Further

$$1 \leq k^s \leq \delta_{k,s} \leq d_{k,s} \leq \gamma_{k,s}, \quad (\text{by (2.19)})$$

implies

$$|k^s + e^{i\theta}| \leq |\delta_{k,s} + e^{i\theta}| \leq |d_{k,s} + e^{i\theta}| \leq |\gamma_{k,s} + e^{i\theta}|, \quad 0 \leq \theta \leq 2\pi,$$

thereby proving last three inequalities in (1.18). This completes the proof of Theorem 2. \square

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Some Fixed Point Theorems for G -Nonexpansive Mappings on Ultrametric Spaces and Non-Archimedean Normed Spaces with a Graph

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ABSTRACT: A very interesting approach in the theory of fixed point in some general structures was recently given by Jachymski by using the context of metric spaces endowed with a graph. The purpose of this article is to present some new fixed point results for G -nonexpansive mappings defined on an ultrametric space and non-Archimedean normed space which are endowed with a graph. In particular, we investigate the relationship between weak connectivity graph and the existence of fixed point for these mappings.

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Keywords and Phrases: Fixed point; Ultrametric space; Non-Archimedean space; Nonexpansive mapping; Connected graph.

1. Introduction and Preliminaries

In 2004, Ran and Reurings [16] started a new research direction in fixed point theory by proving the following Banach-Caccioppoli type principle in ordered metric spaces. Later, in 2005, Nieto and Rodríguez-López [9] proved a modified variant of Ran and Reurings's result by removing the continuity of selfmaps. Notice that the case of decreasing operators is treated by Nieto and Rodríguez-López [10], where some interesting applications to ordinary differential equations with periodic boundary conditions are also given. Also, Nieto, Pouso and Rodríguez-López improved in a recent paper [8] some results on the same topic given by Petrusel and Rus [14] by working in the setting of abstract L-spaces in the sense of Fréchet [8]. Very recently, Agarwal, El-Gebeily and O'Regan extended in [1] the result of Ran and Reurings [16] for the case of generalized ϕ -contractions, while O'Regan and Petrusel [11] proved some new

fixed point results for ϕ -contractions on ordered metric spaces with applications to integral equations. The case of weakly contractive mappings in ordered metric spaces is treated by Harjani and Sadarangani in [5]. A very interesting approach was given by Jachymski [6] and by Gwóźdz-Lukawska and Jachymski [4], where the authors studied the case of self-operators on metric spaces endowed with a graph.

In this paper, we first recall some basic notions in ultrametric spaces and non-Archimedean normed spaces, and motivated by the works of Petalas and Vidalis [12], Kirk and Shazad [7] and Jachymski [6], introduce two new conditions for non-expansive mappings on complete ultrametric spaces (non-Archimedean spaces) and, using these conditions, obtain some fixed point theorems.

The founding father of non-Archimedean functional analysis was Monna, who wrote a series of paper in 1943. A milestone was reached in 1978 at the publication of van Rooij's book [17], the most extensive treatment on non-Archimedean Banach spaces existing in the literature. For more details the reader is referred to [3, 13, 17]. The idea is reasonable to try and generalize ordinary functional analysis by replacing \mathbb{R} and \mathbb{C} by other topological field. This ought to give a new insight in analysis by showing what properties of the scalar field are crucial for certain classical theorems. For this topological field Monna choose a field \mathbb{K} , provided with real valued absolute value function $|\cdot|$ such that \mathbb{K} is complete relative to the metric induced by $|\cdot|$. Adding the condition that, as a topological field, \mathbb{K} is neither \mathbb{R} nor \mathbb{C} , Monna proved the so-called strong triangle inequality

$$|x + y| \leq \max\{|x|, |y|\} \quad (x, y \in \mathbb{K}).$$

This formula is essential to theorems in non-Archimedean functional analysis. Among other things it implies that \mathbb{K} is totally disconnected and cannot be made into a totally ordered field [17].

Van Rooij [17] introduced the concept of ultrametric space as follows:

Let (X, d) be a metric space. (X, d) is called an ultrametric space if the metric d satisfies the strong triangle inequality, i.e., for all $x, y, z \in X$:

$$d(x, y) \leq \max\{d(x, z), d(y, z)\},$$

in this case d is said to be ultrametric. We denote by $B(x, r)$, the closed ball

$$B(x, r) = \{y \in X : d(x, y) \leq r\},$$

where $x \in X$ and we let $r \geq 0$, $B(x, 0) = \{x\}$. A known characteristic property of ultrametric spaces is the following:

$$\text{If } x, y \in X, 0 \leq r \leq s \text{ and } B(x, r) \cap B(y, s) \neq \emptyset, \text{ then } B(x, r) \subset B(y, s).$$

An ultrametric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection. A non-Archimedean valued field is a field \mathbb{K} equipped with a function (valuation) $|\cdot|$ from \mathbb{K} into $[0, \infty)$ such that $|x| = 0$ if and only if $x = 0$, $|x + y| \leq \max\{|x|, |y|\}$ and $|xy| = |x||y|$ for all $x, y \in \mathbb{K}$. Clearly, $|1| = |-1| = 1$ and $|n \cdot 1_{\mathbb{K}}| \leq 1$ for all $n \in \mathbb{N}$ [17].

An example of a non-Archimedean valuation is the mapping $|\cdot|$ taking each point of an arbitrary field but 0 into 1 and $|0| = 0$. This valuation is called trivial. The set $\{|x| : x \in \mathbb{K}, x \neq 0\}$ is a subgroup of the multiplicative group $(0, +\infty)$ and it is called the value group of the valuation. The valuation is called trivial, discrete, or dense accordingly as its value group is $\{1\}$, a discrete subset of $(0, +\infty)$, or a dense subset of $(0, \infty)$, respectively [17]. A norm on a vector space X over a non-Archimedean valued field \mathbb{K} is a map $\|\cdot\|$ from X into $[0, \infty)$ with the following properties:

- 1) $\|x\| \neq 0$ if $x \in X \setminus \{0\}$;
- 2) $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ ($x, y \in X$);
- 3) $\|\alpha x\| = |\alpha|\|x\|$ ($\alpha \in \mathbb{K}, x \in X$).

In 1993, Petalas and Vidalis in [12] presented a generalization of a well-known fixed point theorem for the class of spherically complete non-Archimedean normed spaces, and in 2000 Priess-Crampe and Ribenboim in [15] obtained similar results in ultrametric space, but the proofs of these theorems weren't constructive. In 2012 Kirk and Shahzad in [7] gave more constructive proofs of these theorems and strengthened the conclusions as follow:

Theorem 1.1 ([7]). *Suppose that (X, d) is a spherically complete ultrametric space and $T : X \rightarrow X$ is a nonexpansive mapping (i.e., $d(Tx, Ty) \leq d(x, y)$ for every x and y in X). Then every closed ball of the form*

$$B(x, d(x, Tx)) \quad (x \in X),$$

contains either a fixed point of T or a minimal T -invariant closed ball.

Where a ball $B(x, r)$ is called T -invariant if $T(B(x, r)) \subset B(x, r)$ and is called minimal T -invariant if $B(x, r)$ is T -invariant and $d(u, Tu) = r$ for all $u \in B(x, r)$.

2. Main Results

Let $G = (V(G), E(G))$ be a directed graph. By \tilde{G} we denote the undirected graph obtained from G by ignoring the direction of edges. If x and y are two vertices in a graph G , then a path in G from x to y of length n is a sequence $(x_i)_{i=0}^n$ of $n + 1$ vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \dots, n$, we always suppose that paths are of the shortest length. A graph G is called connected if there is a path between any two vertices and is called weakly connected if \tilde{G} is connected. Subsequently, in this paper X is a complete ultrametric space or non-Archimedean normed space with ultrametric d , Δ is the diagonal of the Cartesian product $X \times X$ and G is a directed graph such that the set $V(G)$ of its vertices coincides with X , the set $E(G)$ of its edges contains Δ and G has no parallel edges. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices. We first give our two results with constructive proofs. In fact, we generalize Kirk and Shahzad's result on nonexpansive mappings on ultrametric spaces and non-Archimedean normed spaces endowed with a graph.

Definition 2.1. Let (X, d) be a metric space endowed with a graph G . We say that a mapping $T : X \rightarrow X$ is G -nonexpansive if

- 1) T preserves the edges of G , i.e., $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$; and
- 2) $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$ with $(x, y) \in E(G)$.

Definition 2.2. Suppose that (X, d) is an ultrametric space endowed with a graph G and $T : X \rightarrow X$ a mapping. We would say that a ball $B(x, r)$ is graphically T -invariant if for any $u \in B(x, r)$ that there exists a path between u and x in \tilde{G} with vertices in $B(x, r)$, we have

$$Tu \in B(x, r).$$

Also, a ball $B(x, r)$ is graphically minimal T -invariant if $Tu \in B(x, r)$ and $d(u, Tu) = r$ for any $u \in B(x, r)$ that there exists a path between u and x in \tilde{G} with vertices in $B(x, r)$.

Theorem 2.3. Let (X, d) be an ultrametric space endowed with a graph G , and G -nonexpansive mapping $T : X \rightarrow X$ satisfies the following conditions:

- (A) There exists an $x_0 \in X$ such that $d(x_0, Tx_0) < 1$;
- (B) If $d(x, Tx) < 1$, then there exists a path in \tilde{G} between x and Tx with vertices in $B(x, d(x, Tx))$;
- (C) If $\{B(x_n, d(x_n, Tx_n))\}$ is a nonincreasing sequence of closed balls in X and for each $n \geq 1$, there exists a path in \tilde{G} between x_n and x_{n+1} with vertices in $B(x_n, d(x_n, Tx_n))$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and there exists $z \in \bigcap_{k=1}^{\infty} B(x_{n_k}, r_{n_k})$ such that for each $k \geq 1$, there exists a path in \tilde{G} between x_{n_k} and z with vertices in $B(x_{n_k}, d(x_{n_k}, Tx_{n_k}))$.

Then for each $x \in X$ with $d(x, Tx) < 1$, the closed ball $B(x, d(x, Tx))$ contains a fixed point of T or a graphically minimal T -invariant ball.

Proof. Let $x \in X$ with $d(x, Tx) < 1$, $r = d(x, Tx)$ and $u \in B(x, r)$ be such that there exists a path $(x_0 = x, x_1, x_2, \dots, x_n = u)$ in \tilde{G} with vertices in $B(x, r)$ between u and x . If $u = x$ then $Tu = Tx \in B(x, r)$, and if not, since $x \neq u$ and the path $(x_0 = x, x_1, \dots, x_n = u)$ has the shortest length, we infer that for each i , $x_{i-1} \neq x_i$. Thus

$$\begin{aligned} d(u, Tu) &\leq \max\{d(u, x), d(x, Tx), d(Tx, Tu)\} \\ &\leq \max\{d(u, x), d(x, Tx), d(Tx, Tx_1), d(Tx_1, Tx_2), \dots, d(Tx_{n-1}, Tu)\} \\ &\leq \max\{d(u, x), d(x, Tx), d(x, x_1), d(x_1, x_2), \dots, d(x_{n-1}, u)\} \\ &= d(x, Tx). \end{aligned}$$

On the other hand, since $B(x, d(x, Tx)) \cap B(u, d(u, Tu)) \neq \emptyset$, we have $B(u, d(u, Tu)) \subset B(x, d(x, Tx))$, so $Tu \in B(x, d(x, Tx))$. This means that

$B(x, d(x, Tx))$ is graphically T -invariant. Now, fix $x_0 \in X$ with $d(x_0, Tx_0) < 1$ and let $x_1 = x_0$, $r_1 = d(x_1, Tx_1)$,

$$E_1 = \{x \in B(x_1, r_1) \mid \text{there is a path in } \tilde{G} \\ \text{between } x \text{ and } x_1 \text{ with vertices in } B(x_1, r_1)\},$$

and

$$\mu_1 = \inf\{d(x, Tx) : x \in E_1\}.$$

Suppose $\{\epsilon_n\}$ is a sequence of positive numbers such that $\lim_{n \rightarrow \infty} \epsilon_n = 0$. If $\mu_1 = r_1$, then the proof is completed because in this case either $r_1 = \mu_1 = 0$ therefore x_1 is a fixed point of T in $B(x_1, r_1)$ or $B(x_1, r_1)$ is graphically minimal T -invariant. Now, let $\mu_1 < r_1$. Choose $x_2 \in B(x_1, r_1)$ such that there exists a path in \tilde{G} between x_1 and x_2 , and

$$r_2 = d(x_2, Tx_2) < \min\{r_1, \mu_1 + \epsilon_1\}.$$

Suppose that by induction x_n is obtained. Put

$$E_n = \{x \in B(x_n, r_n) \mid \text{there is a path in } \tilde{G} \\ \text{between } x \text{ and } x_n \text{ with vertices in } B(x_n, r_n)\},$$

and

$$\mu_n = \inf\{d(x, Tx) : x \in E_n\}.$$

If $r_n = 0$ or $\mu_n = r_n$, using the similar argument as for $n=1$, the proof is complete. Otherwise, choose $x_{n+1} \in B(x_n, r_n)$ such that

$$r_{n+1} = d(x_{n+1}, Tx_{n+1}) < \min\{r_n, \mu_n + \epsilon_n\}.$$

If this process ends after a finite number of steps, then we are done. Otherwise, proceeding in the same manner, we obtain a nonincreasing sequence $\{B(x_n, d(x_n, Tx_n))\}$ of nontrivial closed balls. Since $\{r_n\}$ is nonincreasing, $r := \lim_{n \rightarrow \infty} r_n$ exists. Also, $\{\mu_n\}$ is nondecreasing and bounded above, thus, $\mu := \lim_{n \rightarrow \infty} \mu_n$ also exists. Hence by (C), there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ and $z \in \bigcap_{n=1}^{\infty} B(x_{n_k}, r_{n_k})$ such that for each $k \in \mathbb{N}$ there exists a path in \tilde{G} between x_{n_k} and z with vertices in $B(x_{n_k}, d(x_{n_k}, Tx_{n_k}))$. Since $B(x_{n_k}, r_{n_k})$ is graphically T -invariant for all $k \geq 1$, it follows that $Tz \in B(x_{n_k}, r_{n_k})$, for all $k \geq 1$. Therefore,

$$\begin{aligned} \mu_{n_k+1} &\leq d(z, Tz) \\ &\leq \max\{d(x_{n_k}, z), d(x_{n_k}, Tx_{n_k})\} \\ &\leq r_{n_k}, \end{aligned}$$

for all $k \geq 1$. Thus,

$$\begin{aligned} \mu_{n_k+1} &\leq d(z, Tz) \\ &\leq r \\ &\leq r_{n_k+1} \\ &\leq \mu_{n_k} + \epsilon_{n_k}, \end{aligned}$$

for all $k \geq 1$. Letting $k \rightarrow \infty$, we obtain $d(z, Tz) = r = \mu$. On the other hand, if $x \in B(z, d(z, Tz))$, then $d(x, z) \leq d(z, Tz) \leq r_{n_k}$ for each $k \in \mathbb{N}$. Therefore,

$$d(x, x_{n_k}) \leq \max\{d(x, z), d(x_{n_k}, z)\} \leq r_{n_k},$$

for all $k \geq 1$. Hence, $x \in B(x_{n_k}, r_{n_k})$ for all $k \geq 1$. Now, let $x \in B(z, d(z, Tz))$ and there exists a path between x and z . Thus, there exists a path in $B(z, d(z, Tz))$ between x_{n_k} and x for all $k \geq 1$. Hence $\mu_{n_k} \leq d(x, Tx)$ for all $k \geq 1$. Therefore, for each $k \in \mathbb{N}$, $\mu_{n_k} \leq r_{n_k}$. Hence

$$\inf\{d(x, Tx) : x \in B(z, d(z, Tz))\} = d(z, Tz) = r = \mu.$$

If $r = 0$, then z is a fixed point of T in $B(x, d(x, Tx))$, if not, then the closed ball $B(z, d(z, Tz))$ is graphically minimal T -invariant. Therefore the proof is completed. \square

Corollary 2.4. Let (X, \preceq) be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a spherically complete ultrametric space, and $G = (V(G), E(G))$ is a directed graph with $V(G) = X$ and $E(G) = \{(x, y) \in X \times X : x \preceq y\}$. Suppose also that $T : X \rightarrow X$ is a G -nonexpansive mapping such that (A), (B) and (C) in Theorem 2.3 hold. Then for every $x \in X$ with $d(x, Tx) < 1$, the closed ball $B(x, d(x, Tx))$ contains a fixed point of T or a graphically minimal T -invariant ball.

Remark 2.5. Theorem 2.3 remains valid if the ultrametric space (X, d) is replaced by a spherically complete non-Archimedean normed space $(X, \|\cdot\|)$. Also, note that Corollary 2.4 is valid if the ultrametric space (X, d) is replaced with a non-Archimedean normed space $(X, \|\cdot\|)$.

In the previous theorem, we obtained some results on a closed balls $B(x, d(x, Tx))$ with $d(x, Tx) < 1$. In the following Theorem we obtain these results on every weakly connected ball of the form $B(x, d(x, Tx))$ by adding weak connectivity.

Theorem 2.6. Let (X, d) be a spherically complete ultrametric space endowed with graph G and let $T : X \rightarrow X$ be a G -nonexpansive mapping. Suppose also that for each $x \in X$ the ball $B(x, d(x, Tx))$ is weakly connected. Then for each $z \in X$ the closed ball $B(z, d(z, Tz))$ contains either a fixed point of T or a minimal T -invariant ball.

Proof. Let $x \in X$, and $\Gamma = \{B(y, d(y, Ty)) : y \in B(x, d(x, Tx))\}$. Γ can be partially ordered by set inclusion. Then, using Zorn's Lemma, Γ has a minimal element, say $B(z, d(z, Tz))$. If $B(z, d(z, Tz))$ is singleton, then z is a fixed point of T . If not, we show $B(z, d(z, Tz))$ is minimal T -invariant. It is easy to see that the ball $B(z, d(z, Tz))$ is T -invariant. On the other hand, for each $u \in B(z, d(z, Tz))$, we have

$$\begin{aligned} d(u, Tu) &\leq \max\{d(u, z), d(Tu, z)\} \\ &\leq d(z, Tz), \end{aligned}$$

so $d(u, Tu) = d(z, Tz)$, because if $d(u, Tu) \neq d(z, Tz)$, then $B(u, d(u, Tu)) \subset B(z, d(z, Tz))$ and $B(u, d(u, Tu)) \neq B(z, d(z, Tz))$. This contradicts the minimality of $B(z, d(z, Tz))$. Therefore, $B(z, d(z, Tz))$ is minimal T -invariant. \square

Remark 2.7. Theorem 2.6 remains valid if the ultrametric space (X, d) is replaced by a spherically complete non-Archimedean normed space $(X, \|\cdot\|)$.

3. Examples

In this section, we will give some examples to support our theorems. We also compare the hypotheses of Theorems 2.3 and 2.6 in Examples 2 and 3. In the first example, we present a spherically complete ultrametric space endowed with a weakly connected graph to support Theorem 2.6.

Example 1. Let X be the space c_0 over a non-Archimedean valued field \mathbb{K} with the valuation of \mathbb{K} discrete and choose $\pi \in \mathbb{K}$ with $0 < |\pi| < 1$. Define graph $G = (V(G), E(G))$ by $V(G) = X$ and

$$E(G) = \{(x, y) \in X \times X : \text{either } x = y \\ \text{or there exists exactly one } i \in \mathbb{N} \text{ such that } x_i = y_i\}.$$

Let $B(x, r)$ be an arbitrary closed ball in X and let $y, z \in B(x, r)$. If $y = z$, then (z, y) is a path in \tilde{G} from z to y . Otherwise, we have two cases: Either there exists an $i \in \mathbb{N}$ such that $y_i = z_i$ or not.

case 1. Let there exist $i \in \mathbb{N}$ such that $y_i = z_i$. For each $j \neq i$ we define w_j in the following way:

$$w_j = \begin{cases} y_j + z_j & y_j \neq 0, z_j \neq 0, \\ \pi(y_j + z_j) & \text{either } y_j = 0, \text{ or } z_j = 0, \\ \pi^{n_j} & \text{there exists } n_j \text{ such that } |n_j| < 1, n_{j+1} > n_j. \end{cases}$$

Now, put

$$w = (w_1, w_2, \dots, w_{i-1}, z_i, w_{i+1}, \dots).$$

The process of creating of $\{w_k\}$ shows that for each $j \neq i$, $w_j \neq z_j, y_j$ and $|w_j| < r$. Since $\{n_j\}$ is an increasing sequence $\lim_{j \rightarrow \infty} |\pi^{n_j}| = 0$. On the other hand, $\lim_{j \rightarrow \infty} |z_j + y_j| = 0$ and $\lim_{j \rightarrow \infty} |\pi(z_j + y_j)| = 0$. Thus $w \in c_0$, and also for each j , $|w_j| < r$, so $w \in B(0, r)$.

case 2. Let for each $i \in \mathbb{N}$, $z_i \neq y_i$. Put $w = (z_1, y_2, w_3, w_4, \dots)$, where w_j is defined as case 3.

Then for any case (z, w, y) is a path between z and y with vertices in $B(x, r)$. Therefore, $B(x, r)$ is weakly connected. It is well known when \mathbb{K} is a non-Archimedean valued field with the valuation of \mathbb{K} discrete, c_0 is spherically complete. Therefore, all conditions of Theorem 2.6 hold. On the other hand, if there exists $x_0 \in X$ such that $d(x_0, Tx_0) < 1$ for G -nonexpansive mapping $T : c_0 \rightarrow c_0$, the hypothesis (B) of the Theorem 2.3 hold.

In the following example, we show conditions of Theorem 2.3 are independent of conditions of Theorems 2.6.

Example 2. Let X be the space c_0 over a non-Archimedean valued field \mathbb{K} with the valuation of field \mathbb{K} discrete. Suppose $w \in B(0, 1)$ has exactly one zero coordinate. Define graph G' , with $V(G') = X$, and

$$E(G') = \{(x, y) \in X \times X : x = y \\ \text{or } (x, y) \in E(G), (x, w) \in E(G) \text{ and } (y, w) \in E(G)\}.$$

It is obvious that G' isn't weakly connected. Therefore conditions of Theorem 2.6 do not hold. Now, define $T : X \rightarrow X$ by

$$T(x) = \begin{cases} (x_1, x_2, x_3, \dots), & (x, w) \in E(G), \\ (1 + x_1, 2x_2, 2x_3, \dots), & \text{otherwise.} \end{cases}$$

T is a G' -nonexpansive mapping. It can be readily seen that conditions of Theorem 2.3 hold.

The following example shows that the hypotheses of Theorem 2.6 are independent of the hypotheses of Theorem 2.3.

Example 3. Let X be the space c_0 over a non-Archimedean valued field \mathbb{K} with the valuation of \mathbb{K} discrete. We consider X with the graph G defined in Example 3. Let $e \in \mathbb{K}$ with $|e| > 1$. As we have shown in Example 3, for every G -nonexpansive mapping T the hypotheses of Theorem 2.6 hold. Define $T : X \rightarrow X$ by

$$T(x_1, x_2, \dots) = (e, x_1, x_2, \dots),$$

for each $x \in X$. We have

$$d(x, Tx) = \sup\{|x_1 - e|, |x_2 - x_1|, |x_3 - x_2|, \dots\},$$

so $|x_1 - e| \leq d(x, Tx)$. Since $|x_1 - e| = \max\{|x_1|, |e|\}$ and $|e| \geq 1$, we infer $d(x, Tx) \geq 1$ for all $x \in X$. Hence for each $x \in X$, $d(x, Tx) \geq 1$ and the hypotheses of Theorem 2.3 do not hold.

Remark 3.1. It would be interesting to compare our results with results obtained by Alfuraïdan in [2] that investigated the fixed point theorems for nonexpansive mappings in Archimedean Banach normed spaces.

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On Some L_r -Biharmonic Euclidean Hypersurfaces

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ABSTRACT: In decade eighty, Bang-Yen Chen introduced the concept of biharmonic hypersurface in the Euclidean space. An isometrically immersed hypersurface $x : M^n \rightarrow \mathbb{E}^{n+1}$ is said to be biharmonic if $\Delta^2 x = 0$, where Δ is the Laplace operator. We study the L_r -biharmonic hypersurfaces as a generalization of biharmonic ones, where L_r is the linearized operator of the $(r + 1)$ th mean curvature of the hypersurface and in special case we have $L_0 = \Delta$. We prove that L_r -biharmonic hypersurface of L_r -finite type and also L_r -biharmonic hypersurface with at most two distinct principal curvatures in Euclidean spaces are r -minimal.

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Keywords and Phrases: Linearized operator L_r ; L_r -biharmonic hypersurfaces; L_r -finite type hypersurfaces; r -minimal.

1. Introduction

The concept of biharmonic surfaces in Euclidean space has applications in elasticity and fluid mechanics. In sixty decade, G.B. Airy and J.C. Maxwell have studied the plane elastic problems in terms of the biharmonic equation ([1, 13]). In more general case, the subject of polyharmonic functions was developed by E. Almansi, T. Levi-Civita, M. Nicolaescu. In addition to the differential geometric point of view, biharmonic maps are appeared in PDE theory as solutions of a fourth order strongly elliptic semilinear PDE and in computational geometry as the biharmonic Bezier surfaces.

Clearly, the importance of biharmonic maps will be serious where harmonic maps do not exist. For example, since there exists no harmonic map as $\mathbb{T}^2 \rightarrow \mathbb{S}^2$ (whatever the metrics chosen) in the homotopy class of Brower degree ± 1 , it is important to find a biharmonic map from \mathbb{T}^2 into \mathbb{S}^2 (see in [9]). Obviously, harmonic maps are biharmonic but not vis versa. Biharmonic non-harmonic maps are called *proper-biharmonic*. The variational problem associated to the bienergy functional on the set

of Riemannian metrics on a domain gave rise to the biharmonic stress-energy tensor. This is useful to obtain a new example of proper-biharmonic maps for the study of submanifolds with certain geometric properties, like pseudo-umbilical and parallel submanifolds.

A differential geometric motivation of the subject of biharmonic hypersurfaces is the well-known conjecture of Bang-Yen Chen (in 1987) which says that the biharmonic surfaces in Euclidean 3-spaces are minimal ones. Later on, Dimitrić proved that any biharmonic hypersurface in \mathbb{E}^m with at most two distinct principal curvatures is minimal ([8]). Also, in 1995, Hasanis and Vlachos proved extended Chen's result to the hypersurfaces in Euclidean 4-spaces ([10]). Under the assumption of completeness, Akutagawa and Maeta ([2]) gave a generalization of the result to the global version of Chen's conjecture on biharmonic submanifolds in Euclidean spaces. On the other hand, Dimitrić has found a good relation between the finite type hypersurfaces and biharmonic ones. The theory of finite type hypersurfaces is a well-known subject interested by Chen and also L.J. Alias, S.M.B. Kashani and others. One can see main results in the last chapter of Chen's book ([6]). In [11], Kashani has introduced the notion of L_r -finite type hypersurfaces as an extension of finite type ones in the Euclidean space, which is followed in the doctoral thesis of first author.

The map L_r , as an extension of the Laplacian operator $L_0 = \Delta$, stands for the linearized operator of the first variation of the $(r+1)$ th mean curvature of the hypersurface (see, for instance, [17]). This operator is given by $L_r(f) = \text{tr}(P_r \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_r denotes the r th Newton transformation associated to the second fundamental form of the hypersurface and $\nabla^2 f$ is the hessian of f .

It seems interesting to generalize the definition of biharmonic hypersurface by replacing Δ by L_r . We call these hypersurfaces L_r -biharmonic. Since r -minimal immersions are L_r -biharmonic, one can ask naturally "what about the vice versa?"

In this paper, we study L_r -biharmonic hypersurfaces in the Euclidean space \mathbb{E}^{n+1} . Recently, Aminian and Kashani proved ([5]) the L_r -conjecture for the hypersurfaces with at most two distinct principal curvatures. In this paper, we give an alternative proof of this result by a different method. As our first result on L_r -biharmonic hypersurfaces, we prove that each L_r -biharmonic hypersurface of L_r -finite type in the Euclidean space is r -minimal. Then, we show that any L_r -biharmonic hypersurface in Euclidean space with at most two distinct principal curvatures is r -minimal. The case $r = 0$ (biharmonic hypersurfaces) was studied by Dimitrić, [7]. He proved that, biharmonic hypersurface of finite type or concerning at most two distinct principal curvatures is minimal.

Here are our main results.

Theorem 1.1. *The L_r -biharmonic hypersurfaces of L_r -finite type in Euclidean spaces are r -minimal.*

Theorem 1.2. *The only L_r -biharmonic hypersurfaces of Euclidean spaces \mathbb{E}^{n+1} with at most two distinct principal curvatures are the r -minimal ones ($0 \leq r \leq n-1$).*

Corollary 1.3. Every L_1 -biharmonic surface in \mathbb{E}^3 is flat.

Corollary 1.4. Let M^n be a conformally flat L_r -biharmonic hypersurface of \mathbb{E}^{n+1} , $n > 3$. Then M^n is r -minimal.

After the preliminaries in section 2, in the third section, we prove the main results.

2. Preliminaries

In this section, we introduce some basic notations and facts that will appear along the paper from [19], [4] and [11].

Consider an isometrically immersed hypersurface $x : M^n \rightarrow \mathbb{E}^{n+1}$ in the Euclidean space. We choose a local orthonormal frame $\{e_A\}_{1 \leq A \leq n+1}$ in \mathbb{E}^{n+1} , with dual coframe $\{\omega_A\}_{1 \leq A \leq n+1}$, such that, at each point of M , e_1, \dots, e_n are tangent to M and e_{n+1} is the positively oriented unit normal vector. We shall make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, \dots, \leq n+1; \quad 1 \leq i, j, k, \dots, \leq n.$$

Then the structure equations of \mathbb{E}^{n+1} are given by

$$d\omega_A = \sum_{B=1}^{n+1} \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \quad (1)$$

$$d\omega_{AB} = \sum_{C=1}^{n+1} \omega_{AC} \wedge \omega_{CB}. \quad (2)$$

When restricted to M , we have $\omega_{n+1} = 0$ and

$$0 = d\omega_{n+1} = \sum_{i=1}^n \omega_{n+1i} \wedge \omega_i. \quad (3)$$

By Cartan's lemma, there exist functions h_{ij} such that

$$\omega_{n+1i} = \sum_{j=1}^n h_{ij} \omega_j, \quad h_{ij} = h_{ji}. \quad (4)$$

This gives the second fundamental form of M , as $B = \sum_{i,j} h_{ij} \omega_i \omega_j e_{n+1}$. The mean curvature H is defined by $H = \frac{1}{n} \sum_i h_{ii}$. From (1) - (4) we obtain the structure equations of M (see [19]).

$$d\omega_i = \sum_{j=1}^n \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \quad (5)$$

$$d\omega_{ij} = \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l=1}^n R_{ijkl} \omega_k \wedge \omega_l, \quad (6)$$

and the Gauss equations

$$R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}), \quad (7)$$

where R_{ijkl} denotes the components of the Riemannian curvature tensor of M .

Let h_{ijk} denote the covariant derivative of h_{ij} . We have

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}. \quad (8)$$

Thus, by exterior differentiation of (4), we obtain the Codazzi equation

$$h_{ijk} = h_{ikj}. \quad (9)$$

We choose e_1, \dots, e_n such that

$$h_{ij} = \lambda_i \delta_{ij}. \quad (10)$$

The r th mean curvature H_r of the hypersurface is then defined by

$$\binom{n}{r} H_r = \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \lambda_{i_1} \cdots \lambda_{i_r}. \quad (11)$$

And $H_n = \lambda_1 \cdots \lambda_n$, is called the Gauss-Kronecker curvature of M . A hypersurface with zero $(r+1)$ th mean curvature in \mathbb{R}^{n+1} is called r -minimal. To get more information about r -minimal Euclidean hypersurfaces, the reader is referred to [3, 20].

The classical Newton transformations $P_r : \chi(M) \rightarrow \chi(M)$ are defined inductively by the shape operator S as

$$P_0 = I \quad \text{and} \quad P_r = \binom{n}{r} H_r I - S \circ P_{r-1},$$

for every $r = 1, \dots, n$, where I denotes the identity transformation in $\chi(M)$. Equivalently,

$$P_r = \sum_{j=0}^r (-1)^j \binom{n}{r-j} H_{r-j} S^j.$$

Note that by the Cayley-Hamilton theorem stating that any operator is annihilated by its characteristic polynomial, we have $P_n = 0$.

Since each $P_r(p)$ is also a self-adjoint linear operator on each tangent plane $T_p M$ which commutes with $S(p)$. Indeed, $S(p)$ and $P_r(p)$ can be simultaneously diagonalized: if $\{e_1, \dots, e_n\}$ are the eigenvectors of $S(p)$ corresponding to the eigenvalues $\lambda_1(p), \dots, \lambda_n(p)$, respectively, then they are also the eigenvectors of $P_r(p)$ with corresponding eigenvalues given by

$$\mu_{i,r}(p) = \sum_{i_1 < \dots < i_r, i_j \neq i} \lambda_{i_1}(p) \cdots \lambda_{i_r}(p), \quad (12)$$

for every $1 \leq i \leq n$. We have the following formulae for the Newton transformations, [4].

$$\text{tr}(P_r) = c_r H_r, \quad (13)$$

$$\text{tr}(S \circ P_r) = c_r H_{r+1}, \quad (14)$$

and

$$\text{tr}(S^2 \circ P_{n-1}) = nH_1 H_n, \quad \text{tr}(S^2 \circ P_r) = \binom{n}{r+1} (nH_1 H_{r+1} - (n-r-1)H_{r+2}) \quad (15)$$

for $r = 1, \dots, n-2$, where

$$c_r = (n-r) \binom{n}{r} = (r+1) \binom{n}{r+1}.$$

Associated to each Newton transformation P_r , we consider the second-order linear differential operator $L_r : C^\infty(M) \rightarrow C^\infty(M)$ given by

$$L_r(f) = \text{tr}(P_r \circ \nabla^2 f).$$

Here $\nabla^2 f : \chi(M) \rightarrow \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of f and is given by

$$\langle \nabla^2 f(X), Y \rangle = \langle \nabla_X(\nabla f), Y \rangle,$$

where $X, Y \in \chi(M)$, ∇f is the gradient of f and ∇ is the Levi-Civita connections on M .

Now we recall the definition of an L_r -finite type hypersurface from [11], which is the basic notion of the paper.

Definition 2.1. An isometrically immersed hypersurfaces $x : M^n \rightarrow \mathbb{E}^{n+1}$ is said to be of L_r -finite type if x has a finite decomposition $x = \sum_{i=0}^m x_i$, for some positive integer m satisfying the condition that $L_r x_i = \kappa_i x_i$, $\kappa_i \in \mathbb{R}$, $1 \leq i \leq m$, where $x_i : M^n \rightarrow \mathbb{E}^{n+1}$ are smooth maps, $1 \leq i \leq m$, and x_0 is constant. If all κ_i 's are mutually different, M^n is said to be of L_r -m-type. An L_r -m-type hypersurface is said to be null if some κ_i ; $1 \leq i \leq m$, is zero.

3. L_r -biharmonic hypersurfaces

Consider $x : M^n \rightarrow \mathbb{E}^{n+1}$ a connected orientable hypersurface immersed into the Euclidean space, with the Gauss map N . Then M^n is called a L_r -biharmonic hypersurface if and only if $L_r^2 x = 0$ or equivalently, $L_r(H_{r+1}N) = 0$ (see [4]).

By definition of the L_r -biharmonic hypersurface, it is clear that r -minimal immersions are trivially L_r -biharmonic. By using formula for $L_r^2 x$ of [4] and the considering normal and tangent parts of the L_r -biharmonic condition $L_r^2 x = 0$, one obtains necessary and sufficient conditions for M^n to be L_r -biharmonic in \mathbb{E}^{n+1} , namely

$$L_r H_{r+1} = \binom{n}{r+1} H_{r+1} (nH_1 H_{r+1} - (n-r-1)H_{r+2}) = \text{tr}(S^2 \circ P_r) H_{r+1} \quad (16)$$

and

$$(S \circ P_r)(\nabla H_{r+1}) = -\frac{1}{2} \binom{n}{r+1} H_{r+1} \nabla H_{r+1}. \quad (17)$$

In [7], Dimitrić proved that each biharmonic hypersurface of finite type in a Euclidean space is minimal. In Theorem 1.1, we follow Dimitrić's work and prove that each L_r -biharmonic hypersurface of L_r -finite type in a Euclidean space is r -minimal. Case $r = 0$ corresponds to the classical one.

3.1. Proof of Theorem 1.1

Proof. Let $x : M^n \rightarrow \mathbb{E}^{n+1}$ be an isometrically immersed L_r -biharmonic hypersurface of L_r -finite type in the Euclidean space. Then it has finite decomposition

$$x = x_0 + x_{t_1} + \cdots + x_{t_k}, \quad (18)$$

with $L_r x_0 = 0$, $L_r x_{t_i} = \lambda_{t_i} x_{t_i}$ for nonzero distinct eigenvalues $\lambda_{t_1}, \dots, \lambda_{t_k}$ of L_r . By taking L_r^s of (18) we find

$$0 = L_r^s x = \lambda_{t_1}^s x_{t_1} + \cdots + \lambda_{t_k}^s x_{t_k}, \quad s = 2, 3, \dots \quad (19)$$

Since $\lambda_{t_1}, \dots, \lambda_{t_k}$ are distinct eigenvalues of L_r , system (19) is inconsistent unless $k = 0$. Thus, $x = x_0$, which implies that M is r -minimal. \square

In [6], Chen proved that every biharmonic surface in \mathbb{E}^3 is minimal. Dimitrić ([7]) generalizing Chen's result, proved that any biharmonic hypersurface with at most two distinct principal curvatures is minimal. In Theorem 1.2, we generalize this result and prove that any L_r -biharmonic Euclidean hypersurface with at most two distinct principal curvatures in \mathbb{E}^{n+1} is r -minimal.

Since always exists an open dense subset of M on which the multiplicities of the principal curvatures are locally constant (see Reckziegel [16]), therefore we use the following Lemma locally for the proof of Theorem 1.2.

Lemma 3.1. [15] *Let M be an n -dimensional hypersurface in the Euclidean space \mathbb{E}^{n+1} such that multiplicities of principal curvatures are constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than one, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

3.2. Proof of Theorem 1.2

Proof. Let $x : M^n \rightarrow \mathbb{E}^{n+1}$ be an isometrically immersed L_r -biharmonic Euclidean hypersurface. It is enough to prove that $\mathcal{U} = \{p \in M : \nabla H_{r+1}^2(p) \neq 0\}$, our objective is to show that \mathcal{U} is empty.

In order to prove the Theorem 1.2, we considering three different cases as follows.

Case I: $r = n - 1$.

Case II: $r \neq n - 1$ and the multiplicities are greater than one.

Case III: $r \neq n - 1$ and one of the principal curvatures is simple.

Case I: First, we show that the Gauss-Kronecker curvature of M is constant. By using formulae (16) and (17) on \mathcal{U} we get

$$(SoP_{n-1})\nabla H_n = -\frac{1}{2}H_n\nabla H_n, \quad (20)$$

$$L_{n-1}H_n = nH_1H_n^2. \quad (21)$$

But by the Cayley-Hamilton theorem we have $P_n = 0$, so

$$SoP_{n-1} = H_nI, \quad (SoP_{n-1})\nabla H_n = H_n\nabla H_n,$$

which jointly with (20) yields $\nabla H_n^2 = 0$ on \mathcal{U} , which is a contradiction.

If $H_n \neq 0$, by using (21) we obtain that the mean curvature is constant. By the fact that M has at most two principal curvatures and H, H_n are constant, we get that the principal curvatures are constant, so M is isoparametric. A classical result of B. Segre [18], states that isoparametric hypersurfaces in \mathbb{R}^{n+1} with non zero Gauss-Kronecker curvature are locally isometric to S^n . On the other hand, since S^n is of L_{n-1} -1-type (see [11]), by using Theorem 1.1, we conclude that it is impossible. This finishes the proof of case I.

Case II: Since S^n is of L_{n-1} -1-type (see [11]), therefore, if M^n is totally umbilical, then M^n is a piece of \mathbb{E}^n . Therefore, we assume that M has two distinct principal curvatures of multiplicities q and $n - q$, ($q, n - q > 1$).

Consider $\{e_1, \dots, e_n\}$, to be a local orthonormal frame of principal directions of S on \mathcal{U} such that $Se_i = \lambda_i e_i$ for every $i = 1, \dots, n$. We assume that

$$\lambda_1 = \lambda_2 = \dots = \lambda_q = \lambda, \quad \lambda_{q+1} = \dots = \lambda_n = \mu.$$

Therefore from (12) we have

$$P_{r+1}e_i = \mu_{i,r+1}e_i,$$

with

$$\mu_{i,r+1} = \sum_{i_1 < \dots < i_{r+1}, i_j \neq i} \lambda_{i_1} \dots \lambda_{i_{r+1}}.$$

So, we get

$$\begin{aligned} \mu_{1,r+1} = \dots = \mu_{q,r+1} &= \sum_s \binom{q-1}{s} \binom{n-q}{r+1-s} \lambda^s \mu^{r+1-s}, \\ \mu_{q+1,r+1} = \dots = \mu_{n,r+1} &= \sum_s \binom{q}{s} \binom{n-q-1}{r+1-s} \lambda^s \mu^{r+1-s}. \end{aligned} \quad (22)$$

We obtain from (11) that

$$\binom{n}{r+1} H_{r+1} = \sum_s \binom{q}{s} \binom{n-q}{r+1-s} \lambda^s \mu^{r+1-s}. \quad (23)$$

Since $r \neq n-1$, it follows from the inductive definition of P_{r+1} that (17) is equivalent to

$$P_{r+1}(\nabla H_{r+1}^2) = \frac{3}{2} \binom{n}{r+1} H_{r+1} \nabla H_{r+1}^2 \quad \text{on } \mathcal{U}. \quad (24)$$

Therefore, writing

$$\nabla H_{r+1}^2 = \sum_{i=1}^n \langle \nabla H_{r+1}^2, e_i \rangle e_i, \quad (25)$$

we see that (24) is equivalent to

$$\langle \nabla H_{r+1}^2, e_i \rangle (\mu_{i,r+1} - \frac{3}{2} \binom{n}{r+1} H_{r+1}) = 0 \quad \text{on } \mathcal{U},$$

for every $i = 1, \dots, n$. So, there is no loss of generality, assuming that,

$$\mu_{1,r+1} = \dots = \mu_{q,r+1} = \frac{3}{2} \binom{n}{r+1} H_{r+1}. \quad (26)$$

Let us denote the integral submanifolds through $x \in \mathcal{U}$ corresponding to λ and μ by $\mathcal{U}_1^q(x)$ and $\mathcal{U}_1^{n-q}(x)$ respectively. From Lemma 3.1, we know that λ is constant on $\mathcal{U}_1^q(x)$. (22), (23) and (26) imply that μ is constant on $\mathcal{U}_1^q(x)$. Again by Lemma 3.1, we get that μ is constant on $\mathcal{U}_1^{n-q}(x)$. It now follows from [12], p. 182, Vol. I, that \mathcal{U} is locally isometric to the Riemannian product of the maximal integral manifolds $\mathcal{U}_1^q(x)$ and $\mathcal{U}_1^{n-q}(x)$. Therefore, μ is constant on \mathcal{U} . By the same assertion, we know that λ is constant on \mathcal{U} , so H_{r+1} is constant on \mathcal{U} , which is a contradiction. Hence H_{r+1} is constant on M . If $H_{r+1} \neq 0$, then from (16), we obtain that $\text{tr}(S^2 \circ P_r)$ is constant. By the fact that M has two principal curvatures and H_{r+1} , $\text{tr}(S^2 \circ P_r)$ are constant, we get that the principal curvatures are constant. So, M is isoparametric. The discussion as in the last part of the proof of case I, we get the result in Case II.

Case III: In this case, we suppose that M has two distinct principal curvatures of multiplicities 1 and $n-1$. Assume that $\mathcal{U} \neq \emptyset$ (then we will try to get a contradiction). One can express H_{r+1} as a polynomial in λ (the non simple principal curvature of M) with constant coefficients, after that we express λ as a constant multiple of the simple principal curvature of M . By using Otsuki's Lemma (Lemma 3.1), the structure equations of M , and the fact that M is L_r -biharmonic hypersurface, we get that λ satisfies a polynomial with constant coefficients. So λ is constant, hence H_{r+1} is constant, a contradiction with $\mathcal{U} \neq \emptyset$. Therefore, \mathcal{U} is empty.

Here, is the detailed treatment of the proof.

With the assumption that $\mathcal{U} \neq \emptyset$, consider $\{e_1, \dots, e_n\}$, to be a local orthonormal

frame of principal directions of S on \mathcal{U} such that $Se_i = \lambda_i e_i$ for every $i = 1, \dots, n$. We assume

$$\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = \lambda, \quad \lambda_n = \mu.$$

Therefore we have

$$\begin{aligned} \mu_{1,r+1} = \dots = \mu_{n-1,r+1} &= \binom{n-2}{r+1} \lambda^{r+1} + \binom{n-2}{r} \lambda^r \mu, \\ \mu_{n,r+1} &= \binom{n-1}{r+1} \lambda^{r+1}. \end{aligned} \quad (27)$$

We obtain from (12) that

$$\binom{n}{r+1} H_{r+1} = \binom{n-1}{r+1} \lambda^{r+1} + \binom{n-1}{r} \lambda^r \mu. \quad (28)$$

Since $r \neq n-1$, it follows from the inductive definition of P_{r+1} that (17) is equivalent to

$$P_{r+1}(\nabla H_{r+1}^2) = \frac{3}{2} \binom{n}{r+1} H_{r+1} \nabla H_{r+1}^2 \quad \text{on } \mathcal{U}. \quad (29)$$

Therefore, by the formula

$$\nabla H_{r+1}^2 = \sum_{i=1}^n \langle \nabla H_{r+1}^2, e_i \rangle e_i, \quad (30)$$

we see that (29) is equivalent to

$$\langle \nabla H_{r+1}^2, e_i \rangle \left(\mu_{i,r+1} - \frac{3}{2} \binom{n}{r+1} H_{r+1} \right) = 0 \quad \text{on } \mathcal{U},$$

for every $i = 1, \dots, n$. Hence, for every i such that $\langle \nabla H_{r+1}^2, e_i \rangle \neq 0$ on \mathcal{U} we get

$$\mu_{i,r+1} = \frac{3}{2} \binom{n}{r+1} H_{r+1}. \quad (31)$$

So for the expression ∇H_{r+1}^2 in (30) we consider two subcases.

Subcases 1. $\langle \nabla H_{r+1}^2, e_n \rangle \neq 0$, by using (27) and (31), we obtain that

$$H_{r+1} = \frac{2(n-r-1)}{3n} \lambda^{r+1}. \quad (32)$$

Subcases 2. $\langle \nabla H_{r+1}^2, e_n \rangle = 0$, so on \mathcal{U} we have $\langle \nabla H_{r+1}^2, e_j \rangle \neq 0$ for some $j = 1, \dots, n-1$. By using (27), (31) and the formula of $tr(P_{r+1})$, we obtain that

$$H_{r+1} = \frac{(n-r-1)}{n(-\frac{1}{2}n-r+\frac{1}{2})} \lambda^{r+1}. \quad (33)$$

Both states requires the same calculation, so, we consider just state I.

By Lemma 3.1, let us denote the maximal integral submanifold through $x \in \mathcal{U}$, corresponding to λ by $\mathcal{U}_1^{n-1}(x)$. We write

$$d\lambda = \sum_i \lambda_i \omega_i \quad d\mu = \sum_j \mu_j \omega_j. \quad (34)$$

Then Lemma 3.1 implies that $\lambda_1 = \dots = \lambda_{n-1} = 0$. We can assume that $\lambda > 0$ on \mathcal{U} , then (28) and (32) yields

$$\mu = \frac{r+1-n}{3r+3} \lambda. \quad (35)$$

By means of (8) and (10), we obtain that

$$\sum_k h_{ijk} \omega_k = \delta_{ij} d\lambda_j + (\lambda_i - \lambda_j) \omega_{ij}. \quad (36)$$

We adopt the notational convention that $1 \leq a, b, c, \dots \leq n-1$. From (34) and (36), we have

$$\begin{aligned} h_{ijk} &= 0, & \text{if } i \neq j, & \quad \lambda_i = \lambda_j, \\ h_{aab} &= 0, & h_{aan} &= \lambda_n, \\ h_{nna} &= 0, & h_{nnn} &= \mu_n. \end{aligned} \quad (37)$$

Combining this with (9) and the formula

$$\sum_i h_{ani} \omega_i = dh_{an} + \sum_i h_{in} \omega_{ia} + \sum_i h_{ai} \omega_{in} = (\lambda - \mu) \omega_{an},$$

we obtain from (35)

$$\omega_{an} = \frac{\lambda_n}{\lambda - \mu} \omega_a = \frac{(3r+3)\lambda_n}{(2r+2+n)\lambda} \omega_a. \quad (38)$$

Therefore, we have

$$d\omega_n = \sum_a \omega_{na} \wedge \omega_a = 0.$$

Notice that we may consider λ to be locally a function of the parameter s , where s is the arc length of an orthogonal trajectory of the family of the integral submanifolds corresponding to λ . We may put $\omega_n = ds$.

Thus, for $\lambda = \lambda(s)$, we have

$$d\lambda = \lambda_n ds, \quad \lambda_n = \lambda'(s),$$

so, from (38), we get

$$\omega_{an} = \frac{\lambda_n}{\lambda - \mu} \omega_a = \frac{(3r+3)\lambda'(s)}{(2r+2+n)\lambda} \omega_a. \quad (39)$$

According to the structure equations of \mathbb{E}^{n+1} and (39), we may compute

$$\begin{aligned}
 d\omega_{an} &= \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_{bn} + \omega_{an+1} \wedge \omega_{n+1n} \\
 &= \left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda} \right) \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_b - \lambda\mu\omega_a \wedge ds, \\
 d\omega_{an} &= d \left\{ \frac{(3r+3)\lambda'}{(2r+2+n)\lambda} \omega_a \right\} \\
 &= \left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda} \right)' ds \wedge \omega_a + \left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda} \right) d\omega_a \\
 &= \left\{ - \left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda} \right)' + \left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda} \right)^2 \right\} \omega_a \wedge ds \\
 &\quad + \left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda} \right) \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_b.
 \end{aligned} \tag{40}$$

Then we obtain from two equalities above that

$$\left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda} \right)' - \left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda} \right)^2 - \lambda\mu = 0. \tag{41}$$

Combining (41) with (35), we have

$$\left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda} \right)' - \left(\frac{(3r+3)\lambda'}{(2r+2+n)\lambda} \right)^2 - \left(\frac{r+1-n}{3r+3} \right) \lambda^2 = 0. \tag{42}$$

Let us define a function $\beta(s)$, $s \in (-\infty, +\infty)$ by $\beta = \left(\frac{1}{\lambda} \right)^{\frac{3r+3}{2r+2+n}}$, then (42) reduces to

$$\beta'' = \left(\frac{n-r-1}{3r+3} \right) \beta^{\frac{-r-1-2n}{3r+3}}. \tag{43}$$

Integrating (43), we obtain

$$(\beta')^2 = -\beta^{\frac{2r+2-2n}{3r+3}} + c, \tag{44}$$

where c is the constant of integration.

(44) is equivalent to

$$(\lambda')^2 = - \left(\frac{2+2r+n}{3r+3} \right)^2 \lambda^{\frac{8r+4n+8}{2r+2+n}} + c \left(\frac{2+2r+n}{3r+3} \right)^2 \lambda^{\frac{10r+10+2n}{2r+2+n}}. \tag{45}$$

Now by the definition of $L_r H_{k+1} = \text{tr}(P_r \circ \nabla^2 H_{r+1})$, we compute $L_r H_{r+1}$. So we need to compute $\nabla_{e_a} \nabla H_{r+1}$, $\nabla_{e_n} \nabla H_{r+1}$, $P_r(e_a)$ and $P_r(e_n)$.

From (32) we have

$$\nabla H_{r+1} = \frac{2(r+1)(n-r-1)}{3n} \lambda^r \lambda' e_n. \tag{46}$$

By using (39) and (46) we obtain

$$\begin{aligned}
\nabla_{e_a} \nabla H_{r+1} &= \frac{2(r+1)(n-r-1)}{3n} \lambda^r \lambda' \nabla_{e_a} e_n = \frac{2(r+1)(n-r-1)}{3n} \lambda^r \lambda' \sum_b \omega_{nb}(e_a) e_b \\
&= -\frac{2(n-r-1)(r+1)^2}{n(2r+2+n)} \lambda^{r-1} \lambda'^2 e_a \\
\nabla_{e_n} \nabla H_{r+1} &= \frac{2(r+1)(n-r-1)}{3n} \nabla_{e_n} (\lambda^r \lambda' e_n) \\
&= \frac{2r(r+1)(n-r-1)}{3n} \lambda^{r-1} \lambda'^2 e_n + \frac{2(r+1)(n-r-1)}{3n} \lambda^r \lambda'' e_n.
\end{aligned} \tag{47}$$

By using (27) and (35), we compute $P_r(e_a)$ and $P_r(e_n)$.

$$\begin{aligned}
P_r(e_a) &= \mu_{a,r} e_a = \left(\sum_{i_1 < \dots < i_r, i_j \neq a} \lambda_{i_1} \dots \lambda_{i_r} \right) e_a = \binom{n-2}{r} \frac{2r+3}{3r+3} \lambda^r e_a, \\
P_r(e_n) &= \binom{n-1}{r} \lambda^r e_n.
\end{aligned} \tag{48}$$

From (47) and (48), we get

$$\begin{aligned}
L_r H_{r+1} &= c_r H_{r+1} \left(\frac{(-2r-3)(r+1)(n-r-1)}{n(2r+2+n)} \lambda^{r-2} \lambda'^2 \right. \\
&\quad \left. + \frac{r(r+1)}{n} \lambda^{r-2} \lambda'^2 + \frac{r+1}{n} \lambda^{r-1} \lambda'' \right).
\end{aligned} \tag{49}$$

Since M^n is of L_r -biharmonic hypersurface, hence from (16), we get

$$L_r H_{r+1} = H_{r+1} \operatorname{tr}(S^2 \circ P_r) = H_{r+1} \binom{n-1}{r} \frac{2nr+3n-2r-2r^2}{3r+3} \lambda^{r+2}. \tag{50}$$

Combining (49) and (50), we have

$$\lambda \lambda'' + \left(r + \frac{(-2r-3)(n-r-1)}{2r+2+n} \right) \lambda'^2 - \binom{n-1}{r} \frac{n(2nr+3n-2r-2r^2)}{(r+1)(3r+3)} \lambda^4 = 0. \tag{51}$$

(42) is equivalent to

$$\lambda \lambda'' = \frac{5r+5+n}{2r+2+n} \lambda'^2 + \frac{(2r+2+n)(r+1-n)}{(3r+3)^2} \lambda^4. \tag{52}$$

Thus, putting together (51) and (52) one has

$$\begin{aligned}
&\frac{4r^2+12r-rn-2n+8}{2r+2+n} \lambda'^2 \\
&+ \frac{(2r+2+n)(r+1-n) + 3 \binom{n-1}{r} n(2nr+3n-2r-2r^2)}{(3r+3)^2} \lambda^4 = 0.
\end{aligned} \tag{53}$$

We deduce, using (45), (53) and (32), that H_{r+1} is locally constant on \mathcal{U} , which is a contradiction with the definition of \mathcal{U} . Hence H_{r+1} is constant on M . The discussion as in the last part of the proof of the case I, we get the result. \square

An important consequence of the Theorem is the classification of conformally flat L_r -biharmonic hypersurfaces M^n for $n > 3$.

The dimension of the hypersurface plays an important role in the study of conformally flat Euclidean hypersurfaces. For $n = 2$, the existence of isothermal coordinates means that any Riemannian surface is conformally flat. For $n > 3$, the result of Cartan-Schouten states that a conformally flat hypersurface is characterized with two principal curvatures that one multiplicity at least $n - 1$ (see [14] for more details). This significant fact is crucial in our classification of L_r -biharmonic conformally flat Euclidean hypersurfaces M^n for $n > 3$.

As a simple consequence of Theorem 1.2; case III, we obtain the Corollary 1.4.

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